Mathematics for New Technologies in Finance

Solution sheet 4

Exercise 4.1 (Signatures) Through this exercise, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J\colon \mathcal{C}_0^1(J,E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J,E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0,s],E)$ and $Y \in \mathcal{C}_0^1([s,t],E)$.

(a) Let $X_t = t\mathbf{x} \in \mathbb{R}^d$ for all $t \in [0,1]$. Calculate $\mathbf{Sig}_{[0,1]}(X)$.

(b) Let $X \in \mathcal{C}_0^1([0,T],E)$ and $X_0 = 0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2.$$

(c) Let $X \in \mathcal{C}_0^1([0,1],\mathbb{R})$ s.t. $X_t = \sin(t)$ for all $t \in [0,1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(d) Let $X \in \mathcal{C}_0^1([0,1],\mathbb{R}^2)$ s.t. $X_t = (t, \sin(t))$ for all $t \in [0,1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(e) Let $X \in \mathcal{C}_0^1([0,1],\mathbb{R})$ and $n \in \mathbb{N}$. Calculate $\int_0^1 t^n dX_t$ when

1. $X_t = t$
2. $X_t = \sin(t)$

Solution 4.1

(a)
$$\mathbf{Sig}_{[0,1]}(X) = (1, \mathbf{x}, \frac{\mathbf{x}^{\otimes 2}}{2!}, \cdots). \tag{1}$$

(b) By integration by part, we directly show the equality

$$\int_0^1 u_t^{(1)} du_t^{(2)} + \int_0^1 u_t^{(2)} du_t^{(1)} = \int_0^1 d(u^{(1)} \cdot u^{(2)})_t = u_1^{(1)} \cdot u_1^{(2)} \tag{2}$$

(c)
$$\Big(1, \sin(1), \int_0^1 \sin(t)\cos(t)dt\Big) \tag{3}$$

(d)
$$\Big(1, 1, \sin(1), \frac{1}{2}, \int_0^1 \sin(t)dt, \int_0^1 t\cos(t)dt, \int_0^1 \sin(t)\cos(t)dt\Big) \tag{4}$$

(e) 1.
$$\frac{t^{n+1}}{n+1}\Big|_0^1 \tag{5}$$

2.

$$\begin{aligned}\int_0^1 t^n d\sin(t) &= \sin(t)t^n\Big|_0^1 + \int_0^1 nt^{n-1} d\cos(t) \\ &= \sin(t)t^n\Big|_0^1 + \int_0^1 nt^{n-1} d\cos(t) \\ &= \sin(t)t^n\Big|_0^1 + n\cos(t)t^{n-1}\Big|_0^1 - \int_0^1 n(n-1)t^{n-2} d\sin(t) \\ &= \dots\end{aligned} \tag{6}$$

Exercise 4.2 (Ito's formula) Let W be a Brownian motion on $[0,\infty)$ and define

$$Q^n(W) = \sum_{i=1}^n (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2.$$

(a) Prove that $Q^n(W)$ converges to 1 in $\mathbb{L}^2$. Does the same property hold for a smooth ($\mathcal{C}^1$) function? What does this imply on the regularity of the paths of a Brownian motion W?

(b) Prove the following convergence in $\mathbb{L}^2$ sense

$$\lim_{n\to\infty} \sum_{i=1}^n W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \frac{W_1^2 - 1}{2}$$

(c) Prove that if f is smooth and bounded

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \int_0^t \frac{f''(W_s)}{2} ds.$$

Solution 4.2

(a)

$$\begin{aligned}\mathbb{E}\Big[\Big(\sum_{i=1}^n (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2 - 1\Big)^2\Big] &= \mathrm{Var}(\sum_{i=1}^n (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\ &= \sum_{i=1}^n \mathrm{Var}((W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\ &= n(\mathbb{E}(W_{\frac{1}{n}}^4) - \frac{1}{n^2}) \\ &= n(\frac{3}{n^2} - \frac{1}{n^2}) \to 0 \quad \text{as } n \to \infty\end{aligned}$$

The same property does not hold true for any $\mathcal{C}^1$ process X, since the element $Q_n(X)$ always converges to 0 (pathwise) as $n \to \infty$. This is a consequence of the fact that a $\mathcal{C}^1$ function is in particular of bounded variation on any interval as such $[0,1]$. A notable consequence is that the Brownian motion is not BV (and therefore not $\mathcal{C}^1$) on any interval $[0,T]$.

(b) We notice first that

$$2W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = (W_{\frac{i}{n}} + W_{\frac{i-1}{n}})(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) - (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2 = W_{\frac{i}{n}}^2 - W_{\frac{i-1}{n}}^2 - (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2.$$

(c) Ito's formula, here presented in its integral version, follows naturally from the properties of the Brownian motion outlined in the two previous points. In particular, point (b) implies directly that

$$W_1^2 = 1 + \int_0^1 2W_s \mathrm{d}W_s,$$

which is Ito's formula for $f = (\cdot)^2$ and $t = 1$. The extension to any smooth f is a bit technical, but straightforward. Details can be found, for instance, in Theorem 3.3 of [1].

Exercise 4.3 (Black-Scholes model) Let $\sigma > 0$, $X_t = X_0 \exp\{\sigma W_t - \frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t = \sigma X_t dW_t.$$

(b) Let $K > 0$, calculate

$$C_0 = \mathbb{E}[(X_T - K)_+].$$

(c) Let $K > 0$, calculate

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+].$$

Solution 4.3

(a) One can directly check that

$$\begin{aligned} dX_t &= d[X_0 \exp(\sigma W_t - \frac{\sigma^2 t}{2})] = X_0 d[\exp(\sigma W_t - \frac{\sigma^2 t}{2})] \\ &= X_0 \exp(\sigma W_t - \frac{\sigma^2 t}{2})(\sigma dW_t + \frac{1}{2}\partial_t^2 W_t dt - \frac{\sigma^2 dt}{2}) \\ &= \sigma X_t dW_t. \end{aligned}$$

(b) Applying Black-Scholes formula, we have

$$C_0 = X_0 \Phi(d_1) - K\Phi(d_2) \tag{7}$$

where

$$d_1 = \frac{\log(\frac{X_0}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Proving Black-Scholes formula is elementary, but requires some calculations. One has to notice first that $\sigma W_T - \frac{\sigma^2 T}{2} \sim \mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$. Notably, X_T is a log-normal random variable. A similar variable admits an explicit probability density function $f_{X_T} = f_T$. Hence, we may rewrite

$$\mathbb{E}[(X_T - K)_+] = \int_{\mathbb{R}} (x - K)_+ f_T(x)\mathrm{d}x = \int_K^\infty (x - K) f_T(x)\mathrm{d}x.$$

The rest of the calculations is left as an exercise. It may be useful to recall that, if Φ is the cumulative distribution function of a standard Gaussian random variable, $\Phi(-x) = 1 - \Phi(x)$ for any $x \in \mathbb{R}$.

(c)

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+] = \frac{\partial}{\partial X_0}(X_0\Phi(d_1) - K\Phi(d_2)) = \phi(d_1)$$

Exercise 4.4 (Options' pricing in Black-Scholes and Heston models)

(a) Code option pricing and simulation for European call options and Digital call options in a Black-Scholes model. See exercise notebook 1.

(b) Compare simulations based on BS and Heston model. See exercise notebook 1.

Solution 4.4

(a) See solution notebook 1.

(b) See solution notebook 1.

References

[1] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 1998.