Mathematics for New Technologies in Finance

Solution sheet 7

Exercise 7.1 (Bayesian approach to implied volatility) The Black-Scholes formula provides a relationship between the price of a European Call option $C(K, T)$ and volatility $\sigma(K, T)$ for fixed price of underlying S_0, strike K, and maturity T. It is an important transformation in Finance to calculate from $C(K, T)$ the *implied volatility* $\sigma(K, T)$. Proceed in the following steps:

- Define a Gamma prior on implied volatility.
- Define a likelihood, which predicts the price given an implied volatility.
- Construct a posterior via Bayes formula and sample from it via Langevin dynamics. Interpret the resulting algorithm from the perspective of stochastic gradient descent.

Solution 7.1

- We define the following Gamma prior, $\pi(\sigma) = \frac{\sigma^{\alpha-1}\exp(-\beta\sigma)\beta^{\alpha}}{\Gamma(\alpha)}$, for $\alpha > 0$ and $\beta > 0$.
- Denote the call price calculated using Black-Scholes formula based on the implied volatility(σ) as $C(\sigma)$. For the likelihood, σ is kept fixed, exactly as in Black-Scholes model. We assume that the market price follows a log-normal distribution $\log C(K, T) \sim \mathcal{N}(\log C(\delta), \nu_\epsilon)$, where $\epsilon > 0$ is a fixed parameter. The likelihood reads:

$$\mathcal{L}(C(K, T) \mid \sigma) = \frac{1}{\nu_\epsilon C(T, K)\sqrt{2\pi}} \exp\left(\frac{-(\log(C(K, T) - \log(C(\sigma)))^2}{2\nu_\epsilon^2}\right)$$

- Using Bayes formula, we have the posterior distribution

$$\pi(\sigma \mid C(K, T)) \propto \mathcal{L}(C(K, T) \mid \sigma)\pi(\sigma)$$

- Define learning rate η and noise term $\eta^{1/2}\varepsilon_t$, iterative update implied volatility

$$\sigma_{t+1} = \sigma_t - \eta\nabla(-log(\pi(\sigma \mid C(K, T))) + \eta^{1/2}\varepsilon_t$$

This algorithm leverages Bayesian statistics to estimate the implied volatility of a call option by incorporating prior knowledge (through the Gamma prior) and the observed market price (through the likelihood). Langevin dynamics create a stochastic process that eventually converges to samples from the target posterior distribution. It isn't strictly a descent method, it shares some similarities. The negative log-posterior function acts lie a loss function, and the noise term introduces randomness to explore the space of possible volatilities.

Exercise 7.2 (NN for implied volatility) Recall the calculation of Implied volatility using Bayes formula from Exercise 1. Now we want to calculate the *implied volatility* $\sigma(K, T)$ from $C(K, T)$ using neural network. Proceed in the following steps:

- Define a neural network f^θ which takes as input the option price $C(K, T)$, the current price S_0, the strike price K, and the maturity T. The output will be the implied volatility $\sigma(K, T)$.
- Define a loss function L which calculates the difference between the actual price $C(K, T)$ and $f^\theta(C(K, T), S_0, K, T)$ inserted in the Black-Scholes formula.

- Run a gradient descent.

Solution 7.2 See solution notebook 1.

Exercise 7.3 (Breeden-Litzenberger formula)

(a) Is there always a positive implied volatility σ_{imp} related to the option price? If yes, prove it. Otherwise, on which price interval there is always a positive implied volatility σ_{imp} related to the option price?

(b) Prove the Breeden-Litzenberger formula:

$$\partial_K^2 C(T,K)dK = \mathrm{law}(S_T)(dK).$$

(c) Discretize the Breeden-Litzenberger formula and link it with Butterfly spreads.

Solution 7.3

(a) Since

$$\partial_\sigma C(T,K) = N'(d_1)\sqrt{T} > 0$$

we only need to analyze the boundary:

$$\lim_{\sigma\to 0} C(T,K) = (S_0 - K)_+$$

and

$$\lim_{\sigma\to\infty} C(T,K) = S_0$$

(b) We limit ourselves to a simple case, which is to say assuming that the law of S_T admits a density $f(S,T)$, which is smooth and such that $\lim_{S\to+\infty} f(S,T) = 0$. Under these assumptions,

$$\begin{aligned}\partial_K^2 C(T,K) &= \partial_K^2 \int (S-K)_+ f(S,T)dS \\ &= \partial_K \int_K^\infty -f(S,T)dS = f(K,T).\end{aligned}$$

It should be notice that the same result can be proved at a higher level of generality, even circumventing the assumption on the existence of f (in that case, the equivalence between measures holds true in the weak sense).

(c) Let $K_1 < K_2 < K_3$ Then

$$C(T,K_1) + C(T,K_3) - 2C(T,K_2)$$

is exactly Butterfly spread.

Exercise 7.4 (Dupire formula) Assume the following local volatility model:

$$dS_t = \sigma(t,S_t)S_t dW_t.$$

(a) If $\sigma(t,S_t) = \sigma S_t^\beta$, for which value of β, the market has leverage effect (the volatility increases when the stock price goes down), which is empirically observed.

(b) Let V_t be the fair price of an European payoff $h(S_T)$. Prove the backward Kolmogorov equation:

$$\partial_t V_t + \frac{1}{2}\sigma(S,t)^2 S^2 \partial_{SS}^2 V_t = 0$$

(c) Let f_T^S be the probability density function of S_T, prove the forward Kolmogorov equation (Fokker-Planck equation):

$$\partial_T f(S,T) = \frac{1}{2}\partial_S^2\Big(\sigma(S,T)^2 S^2 f(S,T)\Big)$$

(d) Prove by Fokker-Planck equation the Dupire formula:

$$\sigma^2(K,T) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\partial_K^2 C(T,K)}$$

where $C(T,K)$ is the European call option price of maturity T and strike K.

Solution 7.4

(a) $\beta < 0$

(b) The fair price of an European option is define as $V(t,s) := \mathbb{E}[h(S_T^{t,s})]$, where $S^{t,s}$ is the process with dynamics prescribed above and started at time t in the point s. Assume that this process is $\mathcal{C}^{1,2}$ in its variables. By Ito formula, we have

$$dV(t,S_t) = \partial_t V(t,S_t)dt + \partial_S V(t,S_t)dS_t + \frac{1}{2}\partial_{SS}^2 V(t,S_t)\sigma(t,S_t)^2 S_t^2 dt$$

We notice that, due to the Markov property of S, $V(t,S_t) := \mathbb{E}[h(S_T^{t,s})]\big|_{s=S_t} = \mathbb{E}[h(S_T^{t,s})|\mathcal{F}_t]$, and it is in particular a martingale. As a consequence, the drift term must be 0, which completes the proof.

(c) Since the local volatility model is Markov, we can directly apply the Fokker-Plank equation to it and obtain the result. The Fokker-Plank equation can be derived being the adjoint of the Kolmogorov backward equation.

(d)

$$\begin{aligned}
\partial_T C(T,K) &= \partial_T \int (S-K)_+ f(S,T)dS \\
&= \int (S-K)_+ \partial_T f(S,T)dS \\
&= \int (S-K)_+ \frac{1}{2}\partial_S^2\Big(\sigma(S,T)^2 S^2 f(S,T)\Big)dS \\
&= \frac{1}{2}\left[(S-K)\,\partial_S\left\{\sigma(S,T)^2 S^2 f(S,T)\right\}\right]_{S=K}^{S=\infty} - \frac{1}{2}\int_K^\infty \partial_S\left[\sigma^2(S,T)S^2 f(S,T)\right]dS \\
&= [0-0] - \frac{1}{2}\left[\sigma(S,T)^2 S^2 f(S,T)\right]_{S=K}^{S=\infty} \\
&= \frac{1}{2}\sigma(K,T)^2 K^2 f(K,T) \\
&= \frac{1}{2}\sigma(K,T)^2 K^2 \partial_K^2 C(T,K),
\end{aligned}$$

where we have assumed that $\lim_{S\to+\infty}\partial_S\left\{\sigma(S,T)^2 S^2 f(S,T)\right\} = 0$ and the last step was obtained via Breeden-Litzenberger formula.