Mathematics for New Technologies in Finance

Solution sheet 8

Exercise 8.1 (Variance swap hedging) For given time horizon T, we consider a market composed by a unique risky asset S, whose dynamics reads

$$S_t = S_0 + \int_0^t S_s \sigma_s \mathrm{d}W_s, t \in [0, T],$$

where $S_0 > 0$ and $\sigma = \{\sigma_t\}_{t \in [0,T]}$ is an adapted, strictly positive and bounded process. Assume that the interest rate is constant and equal to 0.The goal of the exercise is price and hedge the random part of a so-called variance swap with maturity T, whose payoff reads as follows:

$$J_T := \frac{1}{T} \int_0^T \sigma_s^2 \mathrm{d}s$$

(a) Using Ito's formula, prove that

$$J_T = \frac{2}{T}[\log(S_0) - \log(S_T)] + \frac{2}{T} \int_0^T \sigma_t \mathrm{d}W_t$$

and verify that

$$\frac{2}{T} \int_0^T \sigma_t \mathrm{d}W_t = \int_0^T \frac{2}{TS_t} \mathrm{d}S_t$$

(b) Deduce from the previous question a replication strategy for an option with maturity T and with payoff $\frac{2}{T} \int_0^T \sigma_t \mathrm{d}W_t$. What is the price of this hedge?

(c) Assume that European call options with payoff $\log(S_T)$ are liquidly traded in the market, and one of them can be sold at price p. How can you replicate and hedge J_T? What is the price of this hedge?

Solution 8.1

(a) We apply Ito's formula to $\log(S_t)$. Then,

$$\log(S_T) = \log(S_0) + \int_0^T \frac{\mathrm{d}S_t}{S_t} - \frac{1}{2} \int_0^T \frac{1}{S_s^2} S_s^2 \sigma_s^2 \mathrm{d}s = \log(S_0) + \int_0^T \sigma_s \mathrm{d}W_s - \frac{T}{2} J_T.$$

This directly implies the former result. The latter is immediate from the dynamics of S.

(b) We can consider the portfolio with 0 initial capital and investment strategy $\Delta_t := \frac{2}{TS_t}$. Then, the value of this portfolio is

$$X_T := 0 + \int_0^T \Delta_t \mathrm{d}S_t = \frac{2}{T} \int_0^T \sigma_t \mathrm{d}W_t.$$

Hence, we have found a strategy that perfectly replicates the the option with maturity T and payoff $\frac{2}{T} \int_0^T \sigma_t \mathrm{d}W_t$. The replication price of this strategy corresponds to the initial capital, which is 0. This is actually a consequence of the fact that the payoff is a martingale (only having a diffusion component), whose starting value is 0.

(c) We combine here the results of the two previous points. In order to replicable J_T, one only needs $\frac{2}{T}\log(S_0)$ in cash, $\frac{2}{T}$ European call options with payoff $\log(S_T)$ (which we can trade by assumption). Finally, by point (b), we know that an option with payoff $\frac{2}{T}\int_0^T \sigma_t \mathrm{d}W_t$, and its replication price is 0. Overall, the cost of the hedging portfolio is exactly

$$\frac{2}{T}[\log(S_0) - p].$$

Exercise 8.2 (Weighted variance swap) We place ourselves in the context of Exercise 1. Given a continuous weight function $w : \mathbb{R}_+ \to \mathbb{R}_+$, we consider a (more general) payoff of the form

$$J_T := \frac{1}{T}\int_0^T \sigma_s^2 w(S_s)\mathrm{d}s$$

(a) Define some $k > 0$ the function F such that $F(x) := \int_k^x \int_k^y \frac{2w(z)}{Tz^2}\mathrm{d}z\mathrm{d}y$. Using Ito's formula, prove that

$$J_T = F(S_T) - F(S_0) - \int_0^T F'(S_t)\mathrm{d}S_t.$$

(b) Is it possible to hedge J_T, without knowing σ, by using cash, Call options, Put options and the risky asset S ? (Hint: you may want to have a look at Carr-Madan formula).

Solution 8.2

(a) The function F is $\mathcal{C}^2$. Hence, we can apply Ito's formula and get

$$F(S_T) = F(S_0) + \int_0^T F'(S_t)\,\mathrm{d}S_t + \frac{1}{2}\int_0^T F''(S_t)\,\sigma_t^2 S_t^2 \mathrm{d}t = F(S_0) + \int_0^T F'(S_t)\,\mathrm{d}S_t + J_T,$$

which directly leads to the result.

(b) It is indeed possible to hedge the payoff J_T. Due to the previous point, we only need an European option with payoff $F(S_T)$ at maturity time T, $F(S_0)$ in cash (short position) and an European option with payoff $\int_0^T F'(S_t)\mathrm{d}S_t$. The latter can be immediately replicated by a portfolio with 0 initial capital and an investment strategy $\Delta_t := F'(S_t)$. Hence, we are left with hedging the option with payoff $F(S_T)$. For this, being an European option only depending on the value at maturity, we invoke Carr–Madan's formula, see [1], to claim it can using call and put options with maturity T.

Exercise 8.3 (Pricing in Heston model) Consider a probability space carrying a two-dimensional Brownian motion $\mathcal{B} = (B, W)^T$, and fix a time horizon $T > 0$. The Heston model prescribes a risky asset which follows the following dynamics

$$\begin{cases} S_t = S_0 + \int_0^t rS_s\mathrm{d}s + \int_0^t S_s\sqrt{v_s}\mathrm{d}B_s, t \geq 0 \\ v_t = v_0 + \int_0^t b(\theta - v_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{v_s}\left(\rho\mathrm{d}B_s + \sqrt{1-\rho^2}\mathrm{d}W_s\right), \end{cases}$$

where b, θ, r and σ are positive constants, while $\rho \in [-1, 1]$ is the correlation coefficient. The aim of this exercise is to price an option of the form $g(S_T)$, which corresponds to compute $\mathbb{E}[g(S_T)]$, where g is assumed to be bounded for simplicity.

(a) Define, for $t \in [0, T]$, $p_t := \mathbb{E}[g(S_T)|\mathcal{F}_t]$. Motivate why p_t is a function of X_t and v_t, in the sense that we can write $p_t = \Psi(t, S_t, v_t)$ for some deterministic function Ψ.

(b) Consider the two-dimensional process $V := (S, v)^T$. Verify that

$$V_t = V_0 + \int_0^t \begin{pmatrix} r - \frac{1}{2}V_s^2 \\ b\left(\theta - V_s^2\right) \end{pmatrix} \mathrm{d}s + \int_0^t \sqrt{V_s^2} \begin{pmatrix} 1 & 0 \\ \sigma\rho & \sigma\sqrt{1-\rho^2} \end{pmatrix} \mathrm{d}\mathcal{B}_s, t \geq 0,$$

where V^i, $i \in 0, 1$ is the ith component of V.

(c) Prove that $\Psi(t, S_t, v_t)$ is a martingale.

(d) Assume that Ψ is smooth in all its variables. Using Ito's formula, derive from the martingale property that Ψ must satisfy the PDE

$$\begin{cases} \frac{\partial \Psi}{\partial t} + \left(r - \frac{1}{2}v\right)\frac{\partial \Psi}{\partial x} + b(\theta - v)\frac{\partial \Psi}{\partial v} + \frac{1}{2}v\frac{\partial^2 \Psi}{\partial x^2} + \frac{\sigma^2}{2}v\frac{\partial^2 \Psi}{\partial v^2} + \sigma\rho v\frac{\partial^2 \Psi}{\partial x \partial v} = 0, \text{ on } [0, T) \times \mathbb{R} \times \mathbb{R}_+ \\ \Psi(T, x, v) = g(x) \end{cases}$$

Solution 8.3

(a) This is a direct consequence of the Markov property of the system (X, v), in the sense that (X, v) is a 2-dimensional Ito process, whose drift and volatility are functions of the current value of X and v only (and do not depend on the whole path).

(b) The computation is straightforward.

(c) We use the fact that

$$\Psi(t, S_t, v_t) = \mathbb{E}[g(S_T)|\mathcal{F}_t].$$

Hence, $\Psi(t, S_t, v_t)$ is the conditional expectation of a bounded random variable, which can be easily checked to be a martingale.

(d) First of all, we use Ito's formula to compute $\mathrm{d}\Psi(t, X_t, v_t)$.

$$\begin{aligned} \mathrm{d}\Psi\left(t, X_t, v_t\right) = & \left(\frac{\partial \Psi}{\partial t} + \left(r - \frac{1}{2}v_t\right)\frac{\partial \Psi}{\partial x} + b\left(\theta - v_t\right)\frac{\partial \Psi}{\partial v}\right)\mathrm{d}t \\ & + \frac{v_t}{2}\left[\frac{\partial^2 \Psi}{\partial x^2} + 2\rho\frac{\partial^2 \Psi}{\partial x \partial v} + \frac{\partial^2 \Psi}{\partial v^2}\right]\mathrm{d}t + \sqrt{v_t}\left(\frac{\partial \Psi}{\partial x} + \sigma\rho\frac{\partial \Psi}{\partial v}\right)\mathrm{d}B_t \\ & + \sigma\sqrt{1-\rho^2}\sqrt{v_t}\frac{\partial \Psi}{\partial v}\mathrm{d}W_t. \end{aligned}$$

Now, since $\Psi(t, S_t, v_t)$ is a martingale and an Ito process, its drift must be equal to 0 with probability one. Now, since X and v can take any value in $\mathbb{R}$ and $\mathbb{R}_+$ respectively, with positive probability, we conclude that the PDE holds true on $[0, T) \times \mathbb{R} \times \mathbb{R}_+$. The terminal condition is immediate since $p_T = g(X_T)$.

Exercise 8.4 (Learning in SABR model) See notebook 1.

Solution 8.4 See solution notebook 1.

References

[1] Peter Carr and Dilip B. Madan. Option valuation using the fast fourier transform. 2:61–73, 2000.