INTERMITTENCY AND DISSIPATION REGULARITY IN TURBULENCE

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ABSTRACT. We lay down a geometric-analytic framework to capture properties of energy dissipation within weak solutions to the incompressible Euler equations. For solutions with spatial Besov regularity, it is proved that the Duchon–Robert distribution has improved regularity in a negative Besov space and, in the case it is a Radon measure, it is absolutely continuous with respect to a suitable Hausdorff measure. This imposes quantitative constraints on the dimension of the, possibly fractal, dissipative set and the admissible structure functions exponents, relating to the phenomenon of "intermittency" in turbulence. As a by-product of the approach, we also recover many known "Onsager singularity" type results.

1. INTRODUCTION

The incompressible Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla q = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \mathbb{T}^d \times (0, T) \tag{E}$$

describe the conservation of momentum and mass for a perfect fluid. By testing (E) with divergence-free test functions, weak solutions can be defined without any reference to the pressure. The latter is a posteriori recovered as the unique zero-mean solution to

$$-\Delta q = \operatorname{div}\operatorname{div}(u \otimes u)$$

By the work of Duchon-Robert [43], weak solutions of (E) in $L^3_{x,t}$ satisfy a local energy balance

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(\left(\frac{|u|^2}{2} + q\right)u\right) = -D \quad \text{in } \mathcal{D}'_{x,t},\tag{1.1}$$

with a distributional source/sink term D, known as the "Duchon–Robert distribution", representing the effect of anomalous dissipation, a central phenomenon in the context of "fully developed turbulence" and the related Kolmogorov [65] and Onsager [77] theories. Motivated by the ubiquity of "intermittency" phenomena in turbulent fluids, we are concerned with geometric/analytic properties of the Duchon–Robert distribution under local regularity assumptions on the weak solution. Our main theorem is the following.

Theorem 1.1 (Dissipation regularity). Assume that $u \in L_t^p B_{p,\infty}^{\sigma}$ is a weak solution to (E) for some $p \in [3,\infty]$ and $\sigma \in (0,1)$, with Duchon-Robert distribution D. Then $D \in B_{\frac{p}{3},\infty}^{\frac{2\sigma}{1-\sigma}-1}$ locally in space-time. If in addition D is a real-valued Radon measure, we have

(i) |D| is absolutely continuous with respect to \mathcal{H}^{γ} for any $\gamma \geq 0$ such that

$$\frac{2\sigma}{1-\sigma} > 1 - \frac{p-3}{p}(d+1-\gamma).$$
(1.2)

(ii) If $D \ge 0$, $\forall K \text{ compact } \exists r_0 > 0 \text{ such that}$

$$D(B_r(x,t)) \lesssim r^{\frac{2\sigma}{1-\sigma} - 1 + \frac{p-3}{p}(d+1)} \qquad \forall (x,t) \in K, \, \forall r < r_0,$$

where the implicit constant depends only on K through local norms of u.

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The assumption that D is a Radon measure is satisfied, for instance, whenever u arises as a strong limit of suitable Navier–Stokes weak solutions in $L^3_{x,t}$. See Section 5.1 for further discussion. This theorem generalizes the results of [35]. More importantly, it lays down a framework which captures fine properties on possibly densely distributed concentration sets of D. Indeed, the arguments from [35] fail to give any non-trivial conclusion as soon as the dissipative set is anywhere dense. To overcome this difficulty a more intrinsic approach is necessary. The uniform regularity of D on space-time balls in (ii) provides an $L^{\infty}_{x,t}$ bound on the upper fractional density of D, which might be relevant in view of the intrinsic "multifractal" nature of turbulence [9,51]. The argument is completely local, and it carries over to the case of general open sets modulo minor technical details (see Section 5.4).

Theorem 1.1 gives quite general, and conjecturally sharp, rigidity results on the Euler equations. They establish the positive side of an intermittent version of the Onsager conjecture, which is a natural extension in view of the breakdown of self-similarity and space-time homogeneity in real turbulent flows. Indeed, a direct consequence is the following intermittency-type statement.

Corollary 1.2 (Intermittency). Let $u \in L^3_{x,t}$ be a weak solutions to (E) with a non-trivial Duchon-Robert measure concentrated on a space-time set S with $\dim_{\mathcal{H}} S = \gamma$. For all $p \in [3, \infty]$ for which there exists $\sigma_p \in (0, 1)$ such that $u \in L^p_t B^{\sigma_p}_{p,\infty}$, it must hold

$$\frac{2\sigma_p}{1 - \sigma_p} \le 1 - \frac{p - 3}{p}(d + 1 - \gamma).$$
(1.3)

Note that as long as the dimension γ is less than d + 1, then for p > 3 this bound implies the regularity index σ_p must be strictly below $\frac{1}{3}$. In particular, a non-trivial and lower dimensional dissipation would necessarily result in a quantitative downward deviation from the Besov $\frac{1}{3}$ regularity for all p > 3, translating into the failure of the Kolmogorov prediction of linear structure functions exponents. See Section 5.1 for elaboration.

Theorem 1.1 is a consequence of the following energy identity for $L^3_{x,t}$ weak solutions.

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Proposition 1.3 (Modified energy identity). Let $u \in L^3_{x,t}$ be a weak solution to (E) with Duchon-Robert distribution D. Let

$$\begin{split} E^{\ell} &:= \frac{|u - u_{\ell}|^2}{2}, \\ Q^{\ell} &:= \left(\frac{|u - u_{\ell}|^2}{2} + (q - q_{\ell})\right)(u - u_{\ell}), \\ R^{\ell} &:= u_{\ell} \otimes u_{\ell} - (u \otimes u)_{\ell}, \\ C^{\ell} &:= (u - u_{\ell}) \cdot \operatorname{div} R^{\ell} + (u - u_{\ell}) \otimes (u - u_{\ell}) : \nabla u_{\ell}, \end{split}$$

with u_{ℓ} the space mollification of u. For all $\ell > 0$, the following identity holds

$$-D = (\partial_t + u_\ell \cdot \nabla) E^\ell + \operatorname{div} Q^\ell + C^\ell \qquad \text{in } \mathcal{D}'_{x,t}.$$

$$(1.4)$$

In Proposition 3.1 the above identity is proved in the more general case of the Navier–Stokes equations. These identities split the dissipation into terms that are small in negative norms, i.e. the ones with E^{ℓ} and Q^{ℓ} , and a term that is large in a positive norm, i.e. C^{ℓ} . By optimizing in the choice of ℓ , we are able to deduce quantitative rates when approximating D with its space-time mollification.

Proposition 1.4 (Mollification rates). Assume that $u \in L_t^p B_{p,\infty}^{\sigma}$ is a weak solution to (E) for some $p \in [3,\infty]$ and $\sigma \in (0,1)$, with Duchon-Robert distribution D. Let ρ_{δ} be a space-time Friedrichs' mollifier. For any $\varphi \in C_{x,t}^{\infty}$ with compact support, there exists $\delta_0 > 0$ such that

$$\left|\left\langle D - D * \rho_{\delta}, \varphi\right\rangle\right| \lesssim \delta^{\frac{2\sigma}{1-\sigma}} \left\|\varphi\right\|_{W^{1,\frac{p}{p-3}}_{x,t}} \qquad and \qquad \left|\left\langle D * \rho_{\delta}, \varphi\right\rangle\right| \lesssim \delta^{\frac{2\sigma}{1-\sigma}-1} \left\|\varphi\right\|_{L^{\frac{p}{p-3}}_{x,t}} \tag{1.5}$$

for all $\delta < \delta_0$, where the implicit constants depend only on norms of u around $\operatorname{Spt} \varphi$.

When cutting D into frequencies shells, the two estimates in (1.5) can be used to control all the pieces in the Fourier space, from which the negative Besov regularity for D is deduced. In a different but equivalent terminology, (1.5) can be read by duality as a quantitative convergence of $D * \rho_{\delta}$ to D in a negative Sobolev norm and a controlled blowup of $D * \rho_{\delta}$ in $L_{x,t}^{\frac{p}{3}}$ respectively. Then, the abstract linear interpolation [7,69] yields to the desired negative fractional regularity.

Although the main purpose of the identity (1.4) in this note, together with its analogue (3.2) for the Navier–Stokes equations, is to prepare the ground for Theorem 1.1, in Section 4 we show how all the main energy-type results in this context follow as almost immediate corollaries. Some of them are well known, some are improved versions of previous results, some others are new. We remark also that the results of this paper could be extended to any system of conservation laws, including the transport equation, a case which we outline in Section 5.5, but also systems like magnetohydrodynamics [1, 48] or compressible fluids [4, 40].

In Section 2 we list the main tools that will be used in this note, in Section 3 we prove our main results while in Section 4 all the corollaries. Finally, Section 5 is dedicated to discussions. These include: the main theoretical background, physical significance of the statements, comparison with previous related works, intermittency for the linear transport equation, sharpness of results and their link with the available convex-integration constructions.

2. Tools

In this section we recall the main tools used in this note.

2.1. Besov spaces and mollification estimates. We define the Besov spaces on \mathbb{R}^N by means of the Littlewood–Paley decomposition (see e.g. [7, Chapter 6]). Let $\phi = \phi(\xi)$ be a smooth function such that

$$\operatorname{Spt} \phi \subset \left\{ \xi \in \mathbb{R}^N : \frac{1}{2} < |\xi| < 2 \right\} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \phi \left(2^{-k} \xi \right) = 1 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

For any $k \in \mathbb{Z}$ we define $\phi_k, \psi \in C^{\infty}$ such that

$$\hat{\phi}_k(\xi) := \phi\left(2^{-k}\xi\right) \quad \text{and} \quad \hat{\psi}(\xi) := 1 - \sum_{k \ge 1} \phi\left(2^{-k}\xi\right),$$
(2.1)

where the symbol $\hat{\cdot}$ denotes the Fourier transform. Since $\hat{\psi} \in C^{\infty}$ with $\operatorname{Spt} \hat{\psi} \subset B_2(0)$, then ψ is a Schwartz function. Consequently, for any $p \in [1, \infty]$, any $\alpha \in \mathbb{R}$ and any tempered distribution f, we define the Besov norm on \mathbb{R}^N by

$$\|f\|_{B^{\alpha}_{p,\infty}} := \|f * \psi\|_{L^{p}} + \sup_{k \ge 1} \left(2^{k\alpha} \|f * \phi_{k}\|_{L^{p}} \right).$$
(2.2)

If $U \subset \mathbb{R}^N$ is an open set, and not necessarily the whole space, we say that f belongs to $B_{p,\infty}^{\alpha}$ "locally inside U" if $\chi f \in B_{p,\infty}^{\alpha}$ on the whole space for any smooth χ with compact support in U.

We recall some classical mollification estimates.

Lemma 2.1. For any function $f : \mathbb{R}^N \to \mathbb{R}$ denote by $f_{\ell} = f * \rho_{\ell}$, where ρ is a Friedrichs' mollifier. Fix $\sigma, \alpha \in (0, 1)$ and $p \in [1, \infty]$. There exist implicit constants independent on ℓ such that

$$\|f - f_\ell\|_{L^p} \lesssim \ell^\sigma \|f\|_{B^\sigma_{p,\infty}},\tag{2.3}$$

$$\|\nabla^n f_\ell\|_{L^p} \lesssim \ell^{\sigma-n} \|f\|_{B^{\sigma}_{p,\infty}} \qquad n \ge 1,$$
(2.4)

$$\|\nabla^{n}(f_{\ell}g_{\ell} - (fg)_{\ell})\|_{L^{p}} \lesssim \ell^{\sigma + \alpha - n} \|f\|_{B^{\sigma}_{rp,\infty}} \|g\|_{B^{\alpha}_{r'p,\infty}} \qquad n \ge 0, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$
(2.5)

From [22, 23, 59] it is known that the pressure enjoys the double regularity

$$\|q\|_{L^{\frac{p}{2}}_{t}B^{2\sigma}_{\frac{p}{2},\infty}} \lesssim \|u\|^{2}_{L^{p}_{t}B^{\sigma}_{p,\infty}}$$
(2.6)

for any $p \in (2, \infty]$ and $\sigma \in (0, 1)$. Then, the following is a consequence of (2.3), (2.4), (2.5) and (2.6).

Corollary 2.2. Let $p \in [3, \infty]$ and $\sigma \in (0, 1)$. Let E^{ℓ} , Q^{ℓ} , R^{ℓ} , C^{ℓ} be the quantities defined in the statement of Proposition 1.3. There exist implicit constants independent on ℓ such that

$$\left|E^{\ell}\right\|_{L^{\frac{p}{2}}_{x,t}} \lesssim \ell^{2\sigma} \left\|u\right\|_{L^{p}_{t}B^{\sigma}_{p,\infty}}^{2}, \qquad (2.7)$$

$$\left\|Q^{\ell}\right\|_{L^{\frac{p}{3}}_{x,t}} \lesssim \ell^{3\sigma} \left\|u\right\|^{3}_{L^{p}_{t}B^{\sigma}_{p,\infty}},\tag{2.8}$$

$$\begin{aligned} \left| R^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} + \ell \left\| \operatorname{div} R^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} \lesssim \ell^{2\sigma} \left\| u \right\|_{L^{p}_{t}B^{\sigma}_{p,\infty}}^{2}, \\ \left\| C^{\ell} \right\|_{L^{\frac{p}{3}}_{x,t}} \lesssim \ell^{3\sigma-1} \left\| u \right\|_{L^{p}_{t}B^{\sigma}_{p,\infty}}^{3}. \end{aligned}$$

$$(2.9)$$

2.2. Inequalities. We recall the following.

Lemma 2.3 (Shinbrot [80, Lemma 4.2]). Let v be a time dependent vector field and f, g be two functions in space-time. Let $\frac{2}{p} + \frac{2}{r} = 1$ with $p \ge 4$. Then

$$\left| \int gv \cdot \nabla f \right| \le \|v\|_{L^r_t L^p_x} \|\nabla f\|_{L^2_{x,t}} \|g\|_{L^\infty_t L^2_x}^{2-\frac{r}{2}} \|g\|_{L^r_t L^p_x}^{\frac{r}{2}-1}.$$

We will need a Young's inequality for convolutions in negative spaces.

Lemma 2.4. On \mathbb{R}^N , let $T \in \mathcal{D}'$ with compact support and $\phi \in S$. For any $p \in [1, \infty]$ it holds

$$||T * \phi||_{L^p} \le ||T||_{W^{-1,p}} ||\phi||_{W^{1,1}}.$$

Proof. Since T has compact support, then $T * \phi \in S$. Let $\varphi \in C_c^{\infty}$ be arbitrary and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Denote by $\tilde{\phi}(x) = \phi(-x)$. The standard Young's convolution inequality implies

$$\left| \int (T * \phi) \varphi \right| = \left| \langle T, \varphi * \tilde{\phi} \rangle \right| \le \|T\|_{W^{-1,p}} \left\| \varphi * \tilde{\phi} \right\|_{W^{1,p'}} \le \|T\|_{W^{-1,p}} \left\| \phi \right\|_{W^{1,1}} \|\varphi\|_{L^{p'}}.$$

The thesis follows by taking the supremum over $\|\varphi\|_{L^{p'}} \leq 1$.

2.3. Radon measures. We follow the presentation from [70]. Let $U \subset \mathbb{R}^N$ be open. A, possibly signed, real-valued Radon measure μ on U is a bounded linear functional over the space of continuous functions with compact support in U. The variation measure of μ , denoted by $|\mu|$, is the non-negative measure such that

$$\langle |\mu|, \varphi \rangle = \int \varphi \, d|\mu| := \sup_{g \in C^0, \, |g| \le \varphi} \langle \mu, g \rangle \qquad \forall \varphi \in C_c^0, \, \varphi \ge 0.$$

Clearly, $|\mu| = \mu$ for any non-negative μ . By the Riesz theorem, a Radon measure μ can be identified with a set function over the Borel subsets of U, and $|\mu|(K) < \infty$ for any compact set $K \subset U$. Given a Borel set $S \subset U$ we denote by $\mu \subseteq S$ the measure defined as

$$\mu \llcorner S(A) := \mu(A \cap S) \qquad \forall A \subset U, A \text{ Borel.}$$

We say that μ is concentrated on S if $\mu \equiv \mu \llcorner S$. By the Hahn decomposition theorem, for any signed measure μ we find two disjoint Borel sets $S_+, S_- \subset U$ such that $\mu \llcorner S_{\pm} \ge 0$ and $S_+ \cup S_- = U$. Consequently, $\mu = \mu \llcorner S_+ - \mu \llcorner S_-$ is a decomposition of μ into a positive and a negative part. It holds $|\mu| = \mu \llcorner S_+ + \mu \llcorner S_-$ as measures. In particular, μ is concentrated on S if and only if $|\mu|(S^c) = 0$.

Let μ and λ be two Borel measures, λ non-negative. We say that μ is absolutely continuous with respect to λ , written as $\mu \ll \lambda$, if $|\mu|(A) = 0$ for all Borel sets A such that $\lambda(A) = 0$.

2.4. Fractal dimensions. Given any $A \subset \mathbb{R}^N$ and $\gamma \ge 0$, for any $\delta > 0$ we set

$$\mathcal{H}^{\gamma}_{\delta}(A) := \inf \left\{ \sum_{i} r_{i}^{\gamma} : A \subset \bigcup_{i} B_{r_{i}}, r_{i} < \delta \text{ for all } i \right\}.$$

Then, the γ -dimensional Hausdorff measure is defined as

$$\mathcal{H}^{\gamma}(A) := \sup_{\delta > 0} \mathcal{H}^{\gamma}_{\delta}(A).$$

This is a non-negative Borel measure on \mathbb{R}^N by the classical Carathéodory construction, and \mathcal{H}^N is equivalent to the N-dimensional Lebesgue measure. The Hausdorff dimension is obtained as

$$\dim_{\mathcal{H}} A := \inf\{\gamma \ge 0 : \mathcal{H}^{\gamma}(A) = 0\}.$$

Similarly, the γ -dimensional upper Minkowski content is defined as

$$\overline{\mathcal{M}}^{\gamma}(A) := \limsup_{r \to 0} \frac{\mathcal{H}^{N}([A]_{r})}{r^{N-\gamma}} \quad \text{with } [A]_{r} = \left\{ x \in \mathbb{R}^{N} \colon \operatorname{dist}(x, A) < r \right\}.$$

Then, the corresponding upper Minkowski dimension is given by

$$\overline{\dim}_{\mathcal{M}} A := \inf\{\gamma \ge 0 : \overline{\mathcal{M}}^{\gamma}(A) = 0\}.$$

3. Proof of the main results

We begin by proving the decomposition (1.4) in the more general case of solutions to the Navier–Stokes equations

$$\begin{cases} \partial_t u^{\nu} + \operatorname{div}(u^{\nu} \otimes u^{\nu}) + \nabla q^{\nu} = \nu \Delta u^{\nu} \\ \operatorname{div} u^{\nu} = 0 \end{cases} \quad \text{in } \mathbb{T}^d \times (0, T).$$
(NS)

The incompressible Euler equations (E) correspond to the case $\nu = 0$. In the viscous setting, for $u^{\nu} \in$ $L_t^2 H_x^1 \cap L_{x,t}^3$, the energy balance reads as

$$(\partial_t - \nu \Delta) \frac{|u^{\nu}|^2}{2} + \operatorname{div} \left(\left(\frac{|u^{\nu}|^2}{2} + q^{\nu} \right) u^{\nu} \right) + \nu |\nabla u^{\nu}|^2 = -D^{\nu} \quad \text{in } \mathcal{D}'_{x,t}.$$
(3.1)

For any L^2 initial datum, solutions are known to exists in the class $u^{\nu} \in L_t^{\infty} L_x^2 \cap L_t^2 H_x^1$. These are known as Leray-Hopf weak solutions [56,68]. Weak solutions for which (3.1) holds with a non-negative distribution D^{ν} are called "suitable weak solutions" in Caffarelli–Kohn–Nirenberg [17] or "dissipative" in Duchon–Robert [43].

Proposition 3.1. Let $\nu \geq 0$. Let $u^{\nu} \in L^3_{x,t}$ be a weak solution to (NS) and let D^{ν} be the associated Duchon-Robert distribution defined by (3.1). If $\nu > 0$ we additionally require $u \in L^2_t H^1_r$. Letting u^{ν}_{ℓ} be the space mollification of u^{ν} , we set

$$\begin{split} E^{\ell,\nu} &:= \frac{|u^{\nu} - u_{\ell}^{\nu}|^{2}}{2}, \\ Q^{\ell,\nu} &:= \left(\frac{|u^{\nu} - u_{\ell}^{\nu}|^{2}}{2} + (q^{\nu} - q_{\ell}^{\nu})\right) (u^{\nu} - u_{\ell}^{\nu}) \\ R^{\ell,\nu} &:= u_{\ell}^{\nu} \otimes u_{\ell}^{\nu} - (u^{\nu} \otimes u^{\nu})_{\ell} \\ C^{\ell,\nu} &:= (u^{\nu} - u_{\ell}^{\nu}) \cdot \operatorname{div} R^{\ell,\nu} + (u^{\nu} - u_{\ell}^{\nu}) \otimes (u^{\nu} - u_{\ell}^{\nu}) : \nabla u_{\ell}^{\nu} \end{split}$$

Denote $\mathcal{E}^{\nu} := D^{\nu} + \nu |\nabla u^{\nu}|^2$. For any $\ell > 0$ we have the identity

$$-\mathcal{E}^{\nu} = (\partial_t + u^{\nu}_{\ell} \cdot \nabla - \nu\Delta) E^{\ell,\nu} + \operatorname{div} Q^{\ell,\nu} + C^{\ell,\nu} + \nu |\nabla u^{\nu}_{\ell}|^2 - 2\nu \nabla u^{\nu} : \nabla u^{\nu}_{\ell} \quad in \ \mathcal{D}'_{x,t}.$$
(3.2)

Note that (3.2) can be equivalently written as

$$-D^{\nu} = (\partial_t + u^{\nu}_{\ell} \cdot \nabla - \nu \Delta) E^{\ell,\nu} + \operatorname{div} Q^{\ell,\nu} + C^{\ell,\nu} + \nu \left| \nabla (u^{\nu}_{\ell} - u^{\nu}) \right|^2.$$
(3.3)

Proof. To lighten the notation, we will suppress the superscript ν on all the quantities. Since we are implicitly assuming the pressure to be the unique zero-average solution to $-\Delta q = \operatorname{div} \operatorname{div}(u \otimes u)$, the Calderón–Zygmund estimates imply $q \in L_{x,t}^{\frac{3}{2}}$. This is enough to justify all the computations below.

Let $\varphi \in C_{x,t}^{\infty}$ be any compactly supported function. By straightforward manipulations on (3.1) we get

$$\begin{split} \langle \mathcal{E}, \varphi \rangle &= \int \frac{|u|^2}{2} (\partial_t + \nu \Delta) \varphi + \left(\frac{|u|^2}{2} + q \right) u \cdot \nabla \varphi \\ &= \int \frac{|u - u_\ell|^2}{2} (\partial_t + u_\ell \cdot \nabla + \nu \Delta) \varphi + \left(\frac{|u - u_\ell|^2}{2} + (q - q_\ell) \right) (u - u_\ell) \cdot \nabla \varphi \\ &+ \underbrace{\int u \cdot \partial_t (u_\ell \varphi) - \varphi u \cdot \partial_t u_\ell - \frac{|u_\ell|^2}{2} (\partial_t + \nu \Delta) \varphi - q_\ell u_\ell \cdot \nabla \varphi}_{I} \\ &+ \underbrace{\int u \cdot u_\ell u \cdot \nabla \varphi - \frac{|u_\ell|^2}{2} u \cdot \nabla \varphi + q_\ell u \cdot \nabla \varphi + q u_\ell \cdot \nabla \varphi + \nu u \cdot u_\ell \Delta \varphi}_{II}. \end{split}$$

Recall that u_{ℓ} satisfies

$$\partial_t u_\ell + \operatorname{div}(u_\ell \otimes u_\ell) + \nabla q_\ell - \nu \Delta u_\ell = \operatorname{div} R^\ell, \qquad R^\ell := u_\ell \otimes u_\ell - (u \otimes u)_\ell.$$
(3.4)

Then, the energy balance of u_{ℓ} reads as

$$(\partial_t - \nu\Delta)\frac{|u_\ell|^2}{2} + \operatorname{div}\left(\left(\frac{|u_\ell|^2}{2} + q_\ell\right)u_\ell\right) + \nu|\nabla u_\ell|^2 = u_\ell \cdot \operatorname{div} R^\ell.$$
(3.5)

Hence, using $u_{\ell}\varphi$ as a test function in the weak formulation of (NS), together with (3.4) and (3.5), we obtain

$$I = \int -u \otimes u : \nabla(u_{\ell}\varphi) - q \operatorname{div}(u_{\ell}\varphi) - \nu u \cdot \Delta(u_{\ell}\varphi) + \varphi u \cdot \left(\operatorname{div}(u_{\ell} \otimes u_{\ell}) + \nabla q_{\ell} - \nu \Delta u_{\ell} - \operatorname{div} R^{\ell}\right) - \int \left(\operatorname{div}\left(\frac{|u_{\ell}|^{2}}{2}u_{\ell}\right) - u_{\ell} \cdot \operatorname{div} R^{\ell} + \nu |\nabla u_{\ell}|^{2}\right) \varphi.$$

We integrate by parts the terms with the Laplacian

$$-\int u \cdot \Delta(u_{\ell}\varphi) + \varphi u \cdot \Delta u_{\ell} = \int \nabla u : u_{\ell} \otimes \nabla \varphi + 2\varphi \nabla u : \nabla u_{\ell} + \nabla u_{\ell} : u \cdot \nabla \varphi$$

Therefore

$$\begin{split} I &= \int -u \otimes u : \nabla u_{\ell} \varphi - u \otimes u : u_{\ell} \otimes \nabla \varphi - q u_{\ell} \cdot \nabla \varphi - q_{\ell} u \cdot \nabla \varphi + \varphi u \cdot \operatorname{div}(u_{\ell} \otimes u_{\ell}) \\ &+ \nu \int \nabla u : u_{\ell} \otimes \nabla \varphi + \nabla u_{\ell} : u \otimes \nabla \varphi \\ &- \int \left((u - u_{\ell}) \cdot \operatorname{div} R^{\ell} + \operatorname{div} \left(\frac{|u_{\ell}|^{2}}{2} u_{\ell} \right) + \nu |\nabla u_{\ell}|^{2} - 2\nu \nabla u : \nabla u_{\ell} \right) \varphi. \end{split}$$

We aim to compute I + II. Note that

$$\int u \cdot u_{\ell} \Delta \varphi = -\int \left(\nabla u : u_{\ell} \otimes \nabla \varphi + \nabla u_{\ell} : u \otimes \nabla \varphi \right),$$
$$-\int \frac{|u_{\ell}|^2}{2} u \cdot \nabla \varphi = \int \varphi u \cdot \nabla \frac{|u_{\ell}|^2}{2}.$$

Thus, since $a \otimes b : c \otimes d = (a \cdot c)(b \cdot d)$ and noticing that all the terms with q and q_{ℓ} cancel out, we achieve

$$I + II = \int \left(-u \otimes u : \nabla u_{\ell} + u \cdot \operatorname{div}(u_{\ell} \otimes u_{\ell}) - \operatorname{div}\left(\frac{|u_{\ell}|^{2}}{2}u_{\ell}\right) + u \cdot \nabla \frac{|u_{\ell}|^{2}}{2} + (u_{\ell} - u) \cdot \operatorname{div} R^{\ell} \right) \varphi$$
$$- \nu \int \left(|\nabla u_{\ell}|^{2} - 2\nabla u : \nabla u_{\ell} \right) \varphi.$$

A direct computation shows that

$$-u \otimes u : \nabla u_{\ell} + u \cdot \operatorname{div}(u_{\ell} \otimes u_{\ell}) - \operatorname{div}\left(\frac{|u_{\ell}|^{2}}{2}u_{\ell}\right) + u \cdot \nabla \frac{|u_{\ell}|^{2}}{2} = -(u - u_{\ell}) \otimes (u - u_{\ell}) : \nabla u_{\ell},$$

uding the proof.

concluding the proof.

We are now ready to prove Proposition 1.4, Theorem 1.1 and Corollary 1.2. The symbol * will always denote the space-time convolution.

Proof of Proposition 1.4. Fix $\varphi \in C_{x,t}^{\infty}$ with compact support and find $\delta_0 > 0$ small enough so that $\langle D * \rho_{\delta}, \varphi \rangle$ is well defined $\forall \delta < \delta_0$. By the identity (1.4) we have

$$\begin{split} |\langle D - D * \rho_{\delta}, \varphi \rangle| &= |\langle D, \varphi - \varphi * \rho_{\delta} \rangle| \\ &\leq \left\| E^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} \left(\left\| \partial_{t} (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-2}}_{x,t}} + \left\| u_{\ell} \right\|_{L^{p}_{x,t}} \left\| \nabla (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-3}}_{x,t}} \right) \\ &+ \left\| Q^{\ell} \right\|_{L^{\frac{p}{3}}_{x,t}} \left\| \nabla (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-3}}_{x,t}} + \left\| C^{\ell} \right\|_{L^{\frac{p}{3}}_{x,t}} \left\| \varphi - \varphi * \rho_{\delta} \right\|_{L^{\frac{p}{p-3}}_{x,t}}. \end{split}$$

Thus, by using (2.3), (2.7), (2.8) and (2.9) we deduce

$$|\langle D - D * \rho_{\delta}, \varphi \rangle| \lesssim \left(\ell^{2\sigma} + \ell^{3\sigma}\right) \|\varphi\|_{W^{1,\frac{p}{p-3}}_{x,t}} + \ell^{3\sigma-1}\delta \|\varphi\|_{W^{1,\frac{p}{p-3}}_{x,t}} \lesssim \delta^{\frac{2\sigma}{1-\sigma}} \|\varphi\|_{W^{1,\frac{p}{p-3}}_{x,t}},$$

where we have chosen $\ell^{1-\sigma} = \delta$ to obtain the last estimate. Similarly

$$\begin{split} |\langle D*\rho_{\delta},\varphi\rangle| &= |\langle D,\varphi*\rho_{\delta}\rangle| \\ &\leq \left\|E^{\ell}\right\|_{L^{\frac{p}{2}}_{x,t}} \left(\left\|\partial_{t}\varphi*\rho_{\delta}\right\|_{L^{\frac{p}{p-2}}_{x,t}} + \left\|u_{\ell}\right\|_{L^{p}_{x,t}} \left\|\nabla\varphi*\rho_{\delta}\right\|_{L^{\frac{p}{p-3}}_{x,t}}\right) \\ &+ \left\|Q^{\ell}\right\|_{L^{\frac{p}{3}}_{x,t}} \left\|\nabla\varphi*\rho_{\delta}\right\|_{L^{\frac{p}{p-3}}_{x,t}} + \left\|C^{\ell}\right\|_{L^{\frac{p}{3}}_{x,t}} \left\|\varphi*\rho_{\delta}\right\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &\lesssim \left(\ell^{2\sigma} + \ell^{3\sigma}\right)\delta^{-1} \left\|\varphi\right\|_{L^{\frac{p}{p-3}}_{x,t}} + \ell^{3\sigma-1} \left\|\varphi\right\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &\lesssim \delta^{\frac{2\sigma}{1-\sigma}-1} \left\|\varphi\right\|_{L^{\frac{p}{p-3}}_{x,t}}, \end{split}$$

where we have chosen again $\ell^{1-\sigma} = \delta$.

By duality, we have proved

$$\|D - D * \rho_{\delta}\|_{W^{-1, \frac{p}{3}}_{x, t}} \lesssim \delta^{\frac{2\sigma}{1 - \sigma}} \quad \text{and} \quad \|D * \rho_{\delta}\|_{L^{\frac{p}{3}}_{x, t}} \lesssim \delta^{\frac{2\sigma}{1 - \sigma} - 1}.$$
(3.6)

Proof of Theorem 1.1. We break the proof down into steps.

STEP 0: Localization.

Since we are working in a bounded time interval, we need to localize. By possibly multiplying D by a smooth compactly supported function, we can think of D as a distribution on the whole space \mathbb{R}^{d+1} with compact support. The presence of an additional smooth multiplicative function does not affect the validity of Proposition 1.4. In particular, the localization of D will satisfy the very same estimates (3.6) on \mathbb{R}^{d+1} .

<u>STEP 1</u>: Proof of $D \in B_{\frac{p}{3},\infty}^{\frac{2\sigma}{1-\sigma}-1}$.

Recall the definition of the Besov norm (2.2). Since $\psi \in S_{x,t}$, the estimate on $D * \psi$ is trivial by using (1.1) directly, i.e. the fact that D is a space-time divergence of an $L_{x,t}^{\frac{p}{3}}$ vector field suffices. We are left to prove

$$\sup_{k \ge 1} \left(2^{\left(\frac{2\sigma}{1-\sigma} - 1\right)k} \left\| D * \phi_k \right\|_{L^{\frac{p}{3}}_{x,t}} \right) < \infty.$$
(3.7)

A direct consequence of (2.1) is

$$\sup_{k\geq 1} \left(\|\phi_k\|_{L^1_{x,t}} + 2^{-k} \|\nabla_{x,t}\phi_k\|_{L^1_{x,t}} \right) < \infty.$$

For any $k \ge 1$ and any $\delta > 0$ we split

$$\|D * \phi_k\|_{L^{\frac{p}{3}}_{x,t}} \le \|(D - D * \rho_{\delta}) * \phi_k\|_{L^{\frac{p}{3}}_{x,t}} + \|(D * \rho_{\delta}) * \phi_k\|_{L^{\frac{p}{3}}_{x,t}}$$

By the Young's inequality for convolutions and the second estimate in (3.6) we deduce

$$\|(D*\rho_{\delta})*\phi_{k}\|_{L^{\frac{p}{3}}_{x,t}} \leq \|D*\rho_{\delta}\|_{L^{\frac{p}{3}}_{x,t}} \|\phi_{k}\|_{L^{1}_{x,t}} \lesssim \delta^{\frac{2\sigma}{1-\sigma}-1}.$$

Similarly, by Lemma 2.4 and the first estimate in (3.6) we get

$$\|(D - D * \rho_{\delta}) * \phi_{k}\|_{L^{\frac{p}{2}}_{x,t}} \leq \|D - D * \rho_{\delta}\|_{W^{-1,\frac{p}{2}}_{x,t}} \|\phi_{k}\|_{W^{1,1}_{x,t}} \lesssim \delta^{\frac{2\sigma}{1-\sigma}} 2^{k}.$$

These estimates together imply

$$\|D*\phi_k\|_{L^{\frac{p}{3}}_{x,t}} \lesssim \delta^{\frac{2\sigma}{1-\sigma}-1} + \delta^{\frac{2\sigma}{1-\sigma}}2^k,$$

from which (3.7) follows by the choice $\delta = 2^{-k}$.

<u>Step 2</u>: Proof of (i).

Let $\gamma \geq 0$ be such that (1.2) holds and let A be a Borel set such that $\mathcal{H}^{\gamma}(A) = 0$. Since D is a Radon measure, we can assume A to be compact without loss of generality. By the Hahn decomposition we find two disjoint sets S_+, S_- such that $D_{\sqcup}S_{\pm} \geq 0$ and $D = D_{\sqcup}S_+ - D_{\bot}S_-$. The goal is to show $D_{\bot}S_{\pm}(A) = D(S_{\pm} \cap A) = 0$. This implies $|D|(A) = D_{\sqcup}S_+(A) + D_{\bot}S_-(A) = 0$, concluding the proof. We will only prove $D(S_+ \cap A) = 0$ since the case of S_- is analogous.

Let $\varepsilon > 0$. Since D is a Radon measure we find an open set $O \supset S_+ \cap A$ such that

$$|D|\left(O\setminus\left(S_{+}\cap A\right)\right)<\varepsilon.$$
(3.8)

By $\mathcal{H}^{\gamma}(S_{+} \cap A) = 0$ we find a finite family of balls $\{B_{r_{i}}\}_{i}$ in \mathbb{R}^{d+1} with $r_{i} < 1$ such that

$$S_{+} \cap A \subset \bigcup_{i} B_{r_{i}} \subset \bigcup_{i} B_{4r_{i}} \subset O, \qquad \sum_{i} r_{i}^{\gamma} < \varepsilon.$$

$$(3.9)$$

On each ball we define a cutoff $\chi^i \in C_{x,t}^{\infty}$ such that

$$0 \le \chi^{i} \le 1, \qquad \chi^{i} |_{B_{r_{i}}} \equiv 1, \qquad \chi^{i} |_{B_{2r_{i}}^{c}} \equiv 0 \quad \text{and} \quad |\nabla_{x,t}\chi^{i}| \le 2r_{i}^{-1}.$$
 (3.10)

Set $\chi := \max_i \chi^i$ pointwise. Clearly χ is Lipschitz continuous with compact support in O and thus D can be applied to it. We split

$$D(S_+ \cap A) = \int_{S_+ \cap A} \chi \, dD = \int_O \chi \, dD - \int_{O \setminus (S_+ \cap A)} \chi \, dD.$$

Since $|\chi| \leq 1$, by (3.8) we get

$$\left| \int_{O \setminus (S_+ \cap A)} \chi \, dD \right| \le |D| \left(O \setminus (S_+ \cap A) \right) < \varepsilon.$$

We are left to estimate $|\langle D, \chi \rangle|$. Note that the identity (1.4) implies $D \equiv 0$ whenever $\sigma > \frac{1}{3}$ (see for instance Corollary 4.1 below). Thus, assume $\sigma \le \frac{1}{3}$. Furthermore, we can also restrict to p > 3, since p = 3 forces

 $\sigma > \frac{1}{3}$ by (1.2). Recall that $B_{b,\infty}^{-\alpha} = (B_{b',1}^{\alpha})^*$ for all $\alpha \in \mathbb{R}, b > 1$ (see for instance [7, Corollary 6.2.8]). By STEP 1 we infer that¹

$$|\langle D,\chi\rangle|\lesssim \|\chi\|_{B^{1-\frac{2\sigma}{1-\sigma}}_{\frac{p}{p-3},1}}\lesssim \|\chi\|_{W^{1-\frac{2\sigma}{1-\sigma}+\kappa,\frac{p}{p-3}}_{x,t}}\,,$$

where $\kappa > 0$ is a small parameter to be fixed later. By the sub-additivity of the $\frac{p}{p-3}$ -th power of the fractional Sobolev norm on the maximum between two functions (see for instance [85, Lemma 2.8]) we deduce²

$$|\langle D, \chi \rangle| \lesssim \left(\sum_i \left\| \chi^i \right\|_{W^{1-\frac{2\sigma}{p-3}}_{x,t} + \kappa, \frac{p}{p-3}}^{\frac{p-3}{p}} \right)^{\frac{p-3}{p}}$$

By interpolating the fractional Sobolev norm, the choice of χ^i made in (3.10) implies

$$\left\|\chi^{i}\right\|_{W^{1-\frac{2\sigma}{1-\sigma}+\kappa,\frac{p}{p-3}}_{x,t}} \leq \left\|\chi^{i}\right\|_{L^{\frac{p}{1-\sigma}-\kappa}_{x,t}}^{\frac{2\sigma}{1-\sigma}-\kappa} \left\|\nabla_{x,t}\chi^{i}\right\|_{L^{\frac{p}{p-3}}_{x,t}}^{1-\frac{2\sigma}{1-\sigma}+\kappa} \lesssim r_{i}^{(d+1)\frac{p-3}{p}-1+\frac{2\sigma}{1-\sigma}-\kappa}$$

where the implicit constant does not depend on i. Thus

$$|\langle D, \chi \rangle| \lesssim \left(\sum_{i} r_i^{\left(\frac{2\sigma}{1-\sigma} - 1 - \kappa\right)\frac{p}{p-3} + d + 1}\right)^{\frac{p-3}{p}}.$$

Since γ satisfies the strict inequality (1.2), we can choose $\kappa = \kappa(\sigma, \gamma, d, p) > 0$ small enough such that

$$\left(\frac{2\sigma}{1-\sigma} - 1 - \kappa\right)\frac{p}{p-3} + d + 1 > \gamma.$$

This, together with (3.9), yields to

$$|\langle D, \chi \rangle| \lesssim \left(\sum_i r_i^{\gamma}\right)^{\frac{p-3}{p}} < \varepsilon^{\frac{p-3}{p}}.$$

The proof is concluded since $\varepsilon > 0$ is arbitrary and p > 3.

<u>Step 3</u>: Proof of (ii).

Let K be a compact set and fix any point $(x,t) \in K$. Pick $\chi^r \in C_{x,t}^{\infty}$ such that

$$0 \le \chi^r \le 1, \qquad \chi^r \big|_{B_r(x,t)} \equiv 1, \qquad \chi^r \big|_{B^c_{2r}(x,t)} \equiv 0 \qquad \text{and} \qquad |\nabla_{x,t}\chi^r| \le 2r^{-1}.$$

This choice of χ^r implies

$$\|\chi^r\|_{L^{\frac{p}{p-3}}_{x,t}} + r \,\|\nabla_{x,t}\chi^r\|_{L^{\frac{p}{p-3}}_{x,t}} \lesssim r^{\frac{p-3}{p}(d+1)}.$$

Since $D \ge 0$, then $\langle D, \chi^r \rangle$ gives an upper bound for $D(B_r(x,t))$. By (1.5), for any $\delta > 0$ we bound

$$\begin{split} \langle D, \chi^r \rangle &\leq |\langle D - D * \rho_{\delta}, \chi^r \rangle| + |\langle D * \rho_{\delta}, \chi^r \rangle| \\ &\lesssim \delta^{\frac{2\sigma}{1-\sigma}} \|\chi^r\|_{W^{1,\frac{p}{p-3}}_{x,t}} + \delta^{\frac{2\sigma}{1-\sigma}-1} \|\chi^r\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &\lesssim \delta^{\frac{2\sigma}{1-\sigma}} r^{\frac{p-3}{p}(d+1)-1} + \delta^{\frac{2\sigma}{1-\sigma}-1} r^{\frac{p-3}{p}(d+1)} \\ &\lesssim r^{\frac{2\sigma}{1-\sigma}-1+\frac{p-3}{p}(d+1)}, \end{split}$$

where the last inequality is obtained by the choice $\delta = r$.

¹We are using the embedding $W^{\alpha+\kappa,b} \subset B^{\alpha}_{b,1}$ for all $\alpha \in \mathbb{R}$, $\kappa > 0$ and $b \in [1, \infty]$. See for instance [7, Theorem 6.2.4]. ²We are assuming κ small enough so that $1 - \frac{2\sigma}{1-\sigma} + \kappa < 1$. This is possible since $\sigma > 0$.

Remark 3.2. The fact that $D \in B_{\frac{p}{3},\infty}^{\frac{2\sigma}{1-\sigma}-1}$ could have been proved directly by the definition of the Besov norm and the splitting (1.4). Indeed, by (1.4) and the Young inequality for convolutions we have

$$\begin{split} \|D*\phi_k\|_{L^{\frac{p}{3}}_{x,t}} &\leq \left\|E^\ell\right\|_{L^{\frac{p}{2}}_{x,t}} \left(\|\partial_t \phi_k\|_{L^{1}_{x,t}} + \|u_\ell\|_{L^{p}_{x,t}} \left\|\nabla\phi_k\|_{L^{1}_{x,t}} \right) + \left\|Q^\ell\right\|_{L^{\frac{p}{3}}_{x,t}} \left\|\nabla\phi_k\|_{L^{1}_{x,t}} + \left\|C^\ell\right\|_{L^{\frac{p}{3}}_{x,t}} \left\|\phi_k\|_{L^{1}_{x,t}} \right) \\ &\lesssim \ell^{2\sigma} 2^k + \ell^{3\sigma-1}. \end{split}$$

Choosing $\ell^{1-\sigma} = 2^{-k}$ yields to the desired bound. Then, the mollification estimates (1.5) become a consequence and no longer the cause of $D \in B_{\frac{p}{2},\infty}^{\frac{2\sigma}{1-\sigma}-1}$. Of course, and not surprisingly, the two are equivalent.

We conclude with the proof of the intermittency corollary.

Proof of Corollary 1.2. Let $\gamma \ge 0$ and assume that D is concentrated on a set S with $\dim_{\mathcal{H}} S = \gamma$. We will prove the contrapositive. Assume there exist $p \in [3, \infty]$ and $\sigma_p \in (0, 1)$ such that

$$\frac{2\sigma_p}{1-\sigma_p} > 1 - \frac{p-3}{p}(d+1-\gamma) \qquad \text{and} \qquad u \in L^p_t B^{\sigma_p}_{p,\infty}.$$

Then, we can find $\tilde{\gamma} > \gamma$ such that the strict inequality stays true with γ replaced by $\tilde{\gamma}$. Since $\dim_{\mathcal{H}} S = \gamma$, it must be $\mathcal{H}^{\tilde{\gamma}}(S) = 0$. Thus, (i) from Theorem 1.1 implies |D|(S) = 0. On the other hand, the assumption $D \equiv D \sqcup S$ implies $|D|(S^c) = 0$. Thus $|D| \equiv 0$.

4. Corollaries

4.1. The Euler equations. Several corollaries directly follow from the splitting (1.4).

Corollary 4.1 (Euler energy conservation [24]). If $u \in L^3_t B^{\sigma}_{3,\infty}$ is a weak solution to (E) for some $\sigma > \frac{1}{3}$, then $D \equiv 0$.

Proof. Since $u \in L^3_{x,t}$ and $q \in L^{\frac{3}{2}}_{x,t}$ we have

$$\left\|E^\ell\right\|_{L^1_{x,t}}+\left\|Q^\ell\right\|_{L^1_{x,t}}\to 0\qquad\text{as }\ell\to 0.$$

Moreover, (2.9) implies $\|C^{\ell}\|_{L^{1}_{x,t}} \lesssim \ell^{3\sigma-1}$, which vanishes as $\ell \to 0$ since $\sigma > \frac{1}{3}$. This shows $D \equiv 0$.

Corollary 4.2 (Kinetic energy regularity [59]). Set

$$e(t) := \frac{1}{2} \int_{\mathbb{T}^d} |u(x,t)|^2 dx.$$

If $u \in L^{\infty}_{t}B^{\sigma}_{3,\infty}$ is a weak solution to (E) for some $\sigma \in (0,1)$, then

$$|e(t) - e(s)| \lesssim |t - s|^{\frac{2\sigma}{1 - \sigma}} \qquad for \ a.e. \ t, s.$$

$$\tag{4.1}$$

Proof. By testing (1.4) with a smooth compactly supported function $\eta = \eta(t)$ we deduce

$$\int e\eta' \, dt = \langle D, \eta \rangle = \int \int_{\mathbb{T}^d} \left(E^{\ell} \eta' - C^{\ell} \eta \right) \, dx \, dt.$$

Thus, by (2.7) and (2.9) we get

$$\left| \int e\eta' \right| \lesssim \|\eta'\|_{L^1_t} \ell^{2\sigma} + \|\eta\|_{L^1_t} \ell^{3\sigma-1},$$

the implicit constant being independent on η . For a.e. t, s (say t > s) we let $\eta \to \mathbb{1}_{[s,t]}$ and obtain $|e(t) - e(s)| \leq \ell^{2\sigma} + |t - s| \ell^{3\sigma-1}$. The choice $\ell^{1-\sigma} = |t - s|$ concludes the proof.

Remark 4.3. If in Corollary 4.2 we further assume³ $u \in C_t^0 L_{w,x}^2$, then the thesis (4.1) can be upgraded to hold for every t, s.

Corollary 4.4 (Minkowski Intermittency [35]). Let $p \in [3, \infty]$, $\sigma \in (0, 1)$ and let $u \in L_t^p B_{p,\infty}^{\sigma}$ be a weak solution to (E) with Duchon-Robert distribution D such that $\dim_{\mathcal{M}} (\operatorname{Spt} D) \leq \gamma$. Then

$$\frac{2\sigma}{1-\sigma} > 1 - \frac{p-3}{p}(d+1-\gamma) \qquad \Longrightarrow \qquad D \equiv 0.$$

Note that Corollary 4.4 does not follow from (i) in Theorem 1.1 since we are not assuming D to be a measure.

Proof. Let $\varphi \in C_{x,t}^{\infty}$ be any compactly supported test function. Denote by $[\operatorname{Spt} D]_{\varepsilon}$ the space-time ε -neighborhood of the set $\operatorname{Spt} D$. Let $\chi^{\varepsilon} \in C_{x,t}^{\infty}$ be such that

$$0 \le \chi^{\varepsilon} \le 1, \qquad \chi^{\varepsilon} \big|_{[\operatorname{Spt} D]_{\varepsilon}} \equiv 1, \qquad \chi^{\varepsilon} \big|_{[\operatorname{Spt} D]_{2\varepsilon}^{c}} \equiv 0 \qquad \text{and} \qquad |\nabla_{x,t} \chi^{\varepsilon}| \le 2\varepsilon^{-1}.$$

Then $\langle D, \varphi \rangle = \langle D, \varphi \chi^{\varepsilon} \rangle$, and by the decomposition (1.4) we infer

$$\begin{split} |\langle D, \varphi \rangle| &\lesssim \left(\left\| E^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} \left(1 + \|u_{\ell}\|_{L^{p}_{x,t}} \right) + \left\| Q^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} + \left\| C^{\ell} \right\|_{L^{\frac{p}{3}}_{x,t}} \right) \|\chi^{\varepsilon}\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &+ \left(\left\| E^{\ell} \right\|_{L^{\frac{p}{2}}_{x,t}} \left(1 + \|u_{\ell}\|_{L^{p}_{x,t}} \right) + \left\| Q^{\ell} \right\|_{L^{\frac{p}{3}}_{x,t}} \right) \|\nabla_{x,t} \chi^{\varepsilon}\|_{L^{\frac{p}{p-3}}_{x,t}} \,, \end{split}$$

where the implicit constant depends only on φ . Let $\alpha > 0$ be a small parameter that will be fixed later. The assumption $\overline{\dim}_{\mathcal{M}}(\operatorname{Spt} D) \leq \gamma$ implies

$$\left\|\chi^{\varepsilon}\right\|_{L^{\frac{p}{p-3}}_{x,t}} + \varepsilon \left\|\nabla_{x,t}\chi^{\varepsilon}\right\|_{L^{\frac{p}{p-3}}_{x,t}} \lesssim \varepsilon^{\frac{p-3}{p}(d+1-\gamma-\alpha)},$$

if ε is sufficiently small. Thus, by (2.7), (2.8) and (2.9) we deduce

$$\begin{split} |\langle D,\varphi\rangle| &\lesssim \left(\ell^{2\sigma} + \ell^{3\sigma} + \ell^{3\sigma-1}\right)\varepsilon^{\frac{p-3}{p}(d+1-\gamma-\alpha)} + \left(\ell^{2\sigma} + \ell^{3\sigma}\right)\varepsilon^{\frac{p-3}{p}(d+1-\gamma-\alpha)-1} \\ &\lesssim \ell^{3\sigma-1}\varepsilon^{\frac{p-3}{p}(d+1-\gamma-\alpha)} + \ell^{2\sigma}\varepsilon^{\frac{p-3}{p}(d+1-\gamma-\alpha)-1}, \end{split}$$

where the implicit constant depends only on norms of φ and u. The choice $\ell^{1-\sigma} = \varepsilon$ yields to

$$|\langle D,\varphi\rangle|\lesssim \varepsilon^{\frac{2\sigma}{1-\sigma}-1+\frac{p-3}{p}(d+1-\gamma-\alpha)}.$$

Since σ, p, γ satisfy an open condition, we find $\alpha > 0$ sufficiently small so that the exponent of ε in the above inequality is positive. The proof is concluded by letting $\varepsilon \to 0$.

4.2. The Navier–Stokes equations. The following result provides a genuinely local version of [41].

Corollary 4.5 (Onsager quasi-singularity [41]). Let $\{u^{\nu}\}_{\nu>0} \subset L^2_t H^1_x$ be a sequence of weak solutions to (NS). Denote by $\mathcal{E}^{\nu} := D^{\nu} + \nu |\nabla u^{\nu}|^2$. Then

$$\sup_{\nu>0} \|u^{\nu}\|_{L^3_t B^{\sigma}_{3,\infty}} < \infty, \, \sigma \ge \frac{1}{3} \qquad \Longrightarrow \qquad |\langle \mathcal{E}^{\nu}, \varphi \rangle| \lesssim \nu^{\frac{3\sigma-1}{1+\sigma}}.$$

where the implicit constant depends only on the $L_t^3 B_{3,\infty}^{\sigma}$ norms of u around $\operatorname{Spt} \varphi$ and norms of φ . In particular, if

$$\liminf_{\nu \to 0} \frac{|\langle \mathcal{E}^{\nu}, \varphi \rangle|}{\nu^{\alpha}} > 0 \qquad for \ some \ \alpha \ge 0,$$

then

$$\liminf_{\nu \to 0} \|u^{\nu}\|_{L^3_t B^{\sigma_{\alpha}+\delta}_{3,\infty}} = \infty \qquad \forall \delta > 0, \, \sigma_{\alpha} := \frac{1+\alpha}{3-\alpha}.$$

 $^{{}^{3}}L^{2}_{w,x}$ denotes the space of L^{2}_{x} functions endowed with the weak topology.

Proof. We write the last two terms in the decomposition (3.2) as

$$\int \left(|\nabla u_{\ell}^{\nu}|^2 - 2\nabla u^{\nu} : \nabla u_{\ell}^{\nu} \right) \varphi = \int \left(|\nabla u_{\ell}^{\nu}|^2 - 2\nabla (u^{\nu} - u_{\ell}^{\nu}) : \nabla u_{\ell}^{\nu} - 2 |\nabla u_{\ell}^{\nu}|^2 \right) \varphi$$
$$= -\int |\nabla u_{\ell}^{\nu}|^2 \varphi + 2 \int (u^{\nu} - u_{\ell}^{\nu}) \cdot \Delta u_{\ell}^{\nu} \varphi + 2 \int (u^{\nu} - u_{\ell}^{\nu}) \otimes \nabla \varphi : \nabla u_{\ell}^{\nu}. \quad (4.2)$$

Since $\{u^{\nu}\}_{\nu>0}$ is bounded in $L^{3}_{t}B^{\sigma}_{3,\infty}$, by (2.7), (2.8), (2.9) and (2.4) we estimate

$$\begin{split} |\langle \mathcal{E}^{\nu}, \varphi \rangle| &\lesssim \left\| E^{\ell, \nu} \right\|_{L^{\frac{3}{2}}_{x, t}} \left(1 + \left\| u_{\ell}^{\nu} \right\|_{L^{3}_{x, t}} \right) + \left\| Q^{\ell, \nu} \right\|_{L^{1}_{x, t}} + \left\| C^{\ell, \nu} \right\|_{L^{1}_{x, t}} \\ &+ \nu \left\| E^{\ell, \nu} \right\|_{L^{\frac{3}{2}}_{x, t}} + \nu \left(\left\| \nabla u_{\ell}^{\nu} \right\|_{L^{2}_{x, t}}^{2} + \left\| u^{\nu} - u_{\ell}^{\nu} \right\|_{L^{2}_{x, t}} \left(\left\| \Delta u_{\ell}^{\nu} \right\|_{L^{2}_{x, t}} + \left\| \nabla u_{\ell}^{\nu} \right\|_{L^{2}_{x, t}} \right) \right) \\ &\lesssim \ell^{2\sigma} + \ell^{3\sigma} + \ell^{3\sigma - 1} + \nu \ell^{2\sigma} + \nu \ell^{2(\sigma - 1)} + \nu \ell^{2\sigma - 1} \\ &\lesssim \ell^{3\sigma - 1} + \nu \ell^{2(\sigma - 1)}. \end{split}$$

The proof is concluded by choosing $\ell^{1+\sigma} = \nu$.

The next result provides a precise physical meaning to the term $C^{\ell,\nu}$ in the decomposition (3.2)

Corollary 4.6 (Four-Fifths Law). Let $\{u^{\nu}\}_{\nu>0} \subset L^2_t H^1_x \cap L^3_{x,t}$ be a sequence of weak solutions to (NS). Assume

$$\sup_{\nu>0} \|u^{\nu}\|_{L^2_t B^{\sigma}_{2,\infty}} < \infty \qquad for \ some \ \sigma > 0. \tag{4.3}$$

Let $\{\ell_{\nu}\}_{\nu>0}$ be any infinitesimal sequence such that

$$\limsup_{\nu \to 0} \frac{\nu}{\ell_{\nu}^{2(1-\sigma)}} = 0.$$
(4.4)

Denote $\mathcal{E}^{\nu} := D^{\nu} + \nu |\nabla u^{\nu}|^2$. Then, for any smooth compactly supported $\eta = \eta(t)$, it holds

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_\nu, \ell_I]} \left| \left\langle \mathcal{E}^\nu + C^{\ell, \nu}, \eta \right\rangle \right| = 0.$$

$$\tag{4.5}$$

Proof. Testing formula (3.2) with a smooth time dependent function η with compact support and writing the last term in (3.2) as in (4.2), we estimate

$$\left| \left\langle \mathcal{E}^{\nu} + C^{\ell,\nu}, \eta \right\rangle \right| \le \left\| E^{\ell,\nu} \right\|_{L^{1}_{x,t}} + \nu \left(\| \nabla u^{\nu}_{\ell} \|^{2}_{L^{2}_{x,t}} + \| u^{\nu} - u^{\nu}_{\ell} \|_{L^{2}_{x,t}} \| \Delta u^{\nu}_{\ell} \|_{L^{2}_{x,t}} \right).$$

Thus, (2.3), (2.4) and (2.7) yield to

$$\left|\left\langle \mathcal{E}^{\nu} + C^{\ell,\nu}, \eta\right\rangle\right| \lesssim \ell^{2\sigma} + \nu \ell^{2(\sigma-1)}.$$

Without loss of generality we can assume $\sigma < 1$. Then

$$\sup_{\ell \in [\ell_{\nu}, \ell_{I}]} \left| \left\langle \mathcal{E}^{\nu} + C^{\ell, \nu}, \eta \right\rangle \right| \lesssim \ell_{I}^{2\sigma} + \frac{\nu}{\ell_{\nu}^{2(1-\sigma)}},$$

and our choice (4.4) of the dissipative length scale implies

$$\limsup_{\nu \to 0} \sup_{\ell \in [\ell_{\nu}, \ell_{I}]} \left| \left\langle \mathcal{E}^{\nu} + C^{\ell, \nu}, \eta \right\rangle \right| \lesssim \ell_{I}^{2\sigma}$$

The proof is concluded by letting $\ell_I \to 0$ since $\sigma > 0$.

Remark 4.7. Corollary 4.6 still holds by weakening the uniform Besov bound to just $L^2_{x,t}$ compactness of the sequence $\{u^{\nu}\}_{\nu>0}$. In this case, (4.4) has to be modified accordingly.

Remark 4.8. From (4.5) we deduce that $C^{\ell,\nu}$ is the term that might cause anomalous dissipation in the inviscid limit. Indeed, if non-vanishing, it keeps the rate of dissipation of order 1 all over the scales in the inertial range $[\ell_{\nu}, \ell_I]$. Thus, in view of the Kolmogorov "Four-Fifths Law", it must be equivalent to third order longitudinal increments, at least asymptotically over the relevant range of scales. To see this, set $\delta_{\ell z} u^{\nu}(x, t) := u^{\nu}(x + \ell z, t) - u^{\nu}(x, t)$ and define the local longitudinal third order structure function by

$$S_{\parallel}^{\nu}(x,t;\ell) := \frac{d(d+2)}{12} \int_{\mathbb{S}^{d-1}} \left(z \cdot \delta_{\ell z} u^{\nu}(x,t) \right)^3 d\mathcal{H}^{d-1}(z).$$

Assuming (4.3) and (4.4), the arguments from [46, 74] prove

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_{\nu}, \ell_I]} \left| \left\langle \frac{S_{\parallel}^{\nu}(\ell)}{\ell} + \mathcal{E}^{\nu}, \eta \right\rangle \right| = 0.$$

Then, Corollary 4.6 implies

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_{\nu}, \ell_I]} \left| \left\langle \frac{S_{\parallel}^{\nu}(\ell)}{\ell} - C^{\ell, \nu}, \eta \right\rangle \right| = 0,$$

giving a precise structural meaning to the term $C^{\ell,\nu}$. See [38] for possible implications for self-regularization of turbulence.

Remark 4.9. If in Corollary 4.6 we strength the assumption (4.3) to

$$\sup_{\nu>0} \|u^{\nu}\|_{L^3_t B^{\sigma}_{3,\infty}} < \infty \qquad for \ some \ \sigma>0,$$

then the thesis holds locally in space-time, that is, one can choose any compactly supported $\varphi \in C_{x,t}^{\infty}$ in (4.5) instead of just $\eta = \eta(t)$. As for Remark 4.7, the $L_{x,t}^3$ compactness of $\{u^{\nu}\}_{\nu>0}$ suffices.

The following corollary is also new, quantifying the relevant scales to capture the whole dissipation.

Corollary 4.10 (Resolved dissipation scales). Let $\{u^{\nu}\}_{\nu>0} \subset L^2_t H^1_x$ be a sequence of weak solutions to (NS) such that $\sup_{\nu>0} \nu \int |\nabla u^{\nu}|^2 < \infty$. Assume

$$\sup_{\nu>0} \|u^{\nu}\|_{L^4_t B^{\sigma}_{4,\infty}} < \infty \qquad for \ some \ \sigma>0$$

Let $\{\ell_{\nu}\}_{\nu>0}$ be any infinitesimal sequence such that

$$\limsup_{\nu \to 0} \frac{\ell_{\nu}^{4\sigma}}{\nu} = 0$$

Let $\eta = \eta(t)$ be non-negative, smooth and compactly supported. Denote $\mathcal{E}^{\nu} := D^{\nu} + \nu |\nabla u^{\nu}|^2$. It holds

$$\limsup_{\nu \to 0} \nu \int \left| \nabla u_{\ell_{\nu}}^{\nu} \right|^2 \eta = 0 \qquad \Longrightarrow \qquad \limsup_{\nu \to 0} \left| \langle \mathcal{E}^{\nu}, \eta \rangle \right| = 0.$$

Proof. By applying (3.2) to $\eta = \eta(t)$ we get

$$|\langle \mathcal{E}^{\nu}, \eta \rangle| \lesssim \left\| E^{\ell, \nu} \eta' \right\|_{L^{1}_{x,t}} + \left\| C^{\ell, \nu} \eta \right\|_{L^{1}_{x,t}} + \nu \int |\nabla u^{\nu}_{\ell}|^{2} \eta + \left(\nu \int |\nabla u^{\nu}_{\ell}|^{2} \eta \right)^{\frac{1}{2}},$$

where we have used that $\sup_{\nu>0} \nu \int |\nabla u^{\nu}|^2 < \infty$ by assumption. Choose $\ell = \ell_{\nu}$. Since $\{u^{\nu}\}_{\nu>0}$ is bounded in $L^2_t B^{\sigma}_{2,\infty}$, we use (2.7) to estimate the first term. Thus, we infer

$$\begin{split} \limsup_{\nu \to 0} |\langle \mathcal{E}^{\nu}, \eta \rangle| &\lesssim \limsup_{\nu \to 0} \left(\ell_{\nu}^{2\sigma} + \left\| C^{\ell_{\nu},\nu} \eta \right\|_{L^{1}_{x,t}} + \nu \int \left| \nabla u_{\ell_{\nu}}^{\nu} \right|^{2} \eta + \left(\nu \int \left| \nabla u_{\ell_{\nu}}^{\nu} \right|^{2} \eta \right)^{\frac{1}{2}} \right) \\ &\leq \limsup_{\nu \to 0} \left\| C^{\ell_{\nu},\nu} \eta \right\|_{L^{1}_{x,t}}. \end{split}$$

Note that so far we have only used that $\{u^{\nu}\}_{\nu>0}$ is bounded in $L^2_t B^{\sigma}_{2,\infty}$. The stronger assumption $L^4_t B^{\sigma}_{4,\infty}$ is needed to handle $\|C^{\ell_{\nu},\nu}\eta\|_{L^1_{n+1}}$. Indeed, we estimate

$$\begin{split} \left\| C^{\ell_{\nu},\nu}\eta \right\|_{L^{1}_{x,t}} &\lesssim \left\| u^{\nu} - u^{\nu}_{\ell_{\nu}} \right\|^{2}_{L^{4}_{x,t}} \left\| \nabla u^{\nu}_{\ell_{\nu}}\sqrt{\eta} \right\|_{L^{2}_{x,t}} + \left\| u^{\nu} - u^{\nu}_{\ell_{\nu}} \right\|_{L^{4}_{x,t}} \left\| \eta^{\frac{3}{4}} \operatorname{div} R^{\ell_{\nu},\nu} \right\|_{L^{\frac{4}{3}}_{x,\tau}} \\ &\lesssim \ell^{2\sigma}_{\nu} \left\| \nabla u^{\nu}_{\ell_{\nu}}\sqrt{\eta} \right\|_{L^{2}_{x,t}} + \ell^{2\sigma}_{\nu} \left(\int |\nabla u^{\nu}|^{2} \eta \right)^{\frac{1}{2}}, \end{split}$$

where the last inequality follows by the commutator estimate (2.5), i.e.

$$\left\| \operatorname{div} R^{\ell_{\nu},\nu} \right\|_{L^{\frac{4}{3}}_{x}} \lesssim \ell_{\nu}^{\sigma} \left\| u^{\nu} \right\|_{B^{\sigma}_{4,\infty}} \left\| \nabla u^{\nu} \right\|_{L^{2}_{x}}$$

We conclude

$$\limsup_{\nu \to 0} |\langle \mathcal{E}^{\nu}, \eta \rangle| \lesssim \limsup_{\nu \to 0} \left(\left(\frac{\ell_{\nu}^{4\sigma}}{\nu} \nu \int \left| \nabla u_{\ell_{\nu}}^{\nu} \right|^{2} \eta \right)^{\frac{1}{2}} + \left(\frac{\ell_{\nu}^{4\sigma}}{\nu} \nu \int \left| \nabla u^{\nu} \right|^{2} \eta \right)^{\frac{1}{2}} \right) = 0.$$

Remark 4.11. When $\sigma = \frac{1}{3}$, both the scales from Corollary 4.6 and Corollary 4.10 give $\ell_{\nu} \sim \nu^{\frac{3}{4}}$, i.e. the Kolmogorov dissipative length scale.

Corollary 4.12 (Shinbrot local energy balance [80]). Let $u^{\nu} \in L_t^{\infty} L_x^2 \cap L_t^2 H_x^1 \cap L_{x,t}^3$ be a weak solution to (NS). Assume $u^{\nu} \in L_t^m L_x^p$ for some m, p such that $\frac{2}{p} + \frac{2}{m} \leq 1, p \geq 4$. Then (3.1) holds with $D^{\nu} \equiv 0$.

Proof. Since viscosity is fixed, we omit the superscripts ν . Since $u \in L^3_{x,t} \cap L^2_t H^1_x$ and $q \in L^{\frac{3}{2}}_{x,t}$, we have

$$\left\| E^{\ell} \right\|_{L^{\frac{3}{2}}_{x,t}} + \left\| Q^{\ell} \right\|_{L^{1}_{x,t}} + \left\| \nabla (u_{\ell} - u) \right\|_{L^{2}_{x,t}} \to 0 \qquad \text{as } \ell \to 0.$$

Thus, by (3.3), $D \equiv 0$ whenever $\|C^{\ell}\|_{L^{1}_{x,t}} \to 0$ as $\ell \to 0$. Recall that

$$C^{\ell} = (u - u_{\ell}) \cdot \operatorname{div} R^{\ell} + (u - u_{\ell}) \otimes (u - u_{\ell}) : \nabla u_{\ell} =: C_1^{\ell} + C_2^{\ell}$$

By Lemma 2.3 we estimate

$$\left\|C_{2}^{\ell}\right\|_{L_{x,t}^{1}} \leq \left\|u - u_{\ell}\right\|_{L_{t}^{m}L_{x}^{p}} \left\|\nabla u_{\ell}\right\|_{L_{x,t}^{2}} \left\|u - u_{\ell}\right\|_{L_{t}^{\infty}L_{x}^{2}}^{2-\frac{m}{2}} \left\|u - u_{\ell}\right\|_{L_{t}^{m}L_{x}^{p}}^{\frac{m}{2}-1}.$$
(4.6)

Since we are working locally, by possibly reducing m we can assume $\frac{2}{p} + \frac{2}{m} = 1$. In particular, since $p \ge 4$, it must be $m < \infty$. Thus, it is clear that the expression in the right hand side of (4.6) goes to zero as $\ell \to 0$, if $p < \infty$, by standard properties of mollifiers. If $p = \infty$, then m = 2 necessarily. Thus $u \in L_t^{\infty} L_x^2 \cap L_t^2 L_x^{\infty} \subset L_{x,t}^4$ and we can conclude as before. Moreover, since $u \in L_t^2 H_x^1$, the term C_1^{ℓ} can be written as

$$C_1^{\ell} = (u - u_{\ell}) \otimes u_{\ell} : \nabla u_{\ell} + (u - u_{\ell}) \cdot (u \cdot \nabla u)_{\ell}.$$

Thus, the very same argument we have given for C_2^{ℓ} applies to show $\lim_{\ell \to 0} \left\| C_1^{\ell} \right\|_{L^{1}_{-\ell}} = 0.$

Corollary 4.13 (Uniform bound viscous dissipation). Let $\{u^{\nu}\}_{\nu>0} \subset L^2_t H^1_x$ be a sequence of weak solutions to (NS). Denote by $\mathcal{E}^{\nu} := D^{\nu} + \nu |\nabla u^{\nu}|^2$. If $\{u^{\nu}\}_{\nu>0}$ is bounded in $L^p_t B^{\sigma}_{p,\infty}$ for some $p \in [3,\infty]$ and $\sigma > 0$, then for any $\varphi \in C^{\infty}_{x,t}$ compactly supported $\exists \delta_0 > 0$ such that

$$|\langle \mathcal{E}^{\nu} - \mathcal{E}^{\nu} * \rho_{\delta}, \varphi \rangle| \lesssim \delta^{2\sigma} \left\|\varphi\right\|_{W^{\frac{2}{2}, \frac{p}{p-3}}_{x,t}} \quad and \quad |\langle \mathcal{E}^{\nu} * \rho_{\delta}, \varphi \rangle| \lesssim \delta^{2\sigma-2} \left\|\varphi\right\|_{L^{\frac{p}{p-3}}_{x,t}} \quad \forall \delta < \delta_{0}, \tag{4.7}$$

with implicit constants that are uniform in viscosity. In particular, $\{\mathcal{E}^{\nu}\}_{\nu>0}$ is bounded in $B^{2(\sigma-1)}_{\frac{p}{3},\infty}$ in spacetime, locally. *Proof.* Use the identity (3.2) with the last two terms computed as in (4.2) to bound

$$\begin{split} |\langle \mathcal{E}^{\nu} - \mathcal{E}^{\nu} * \rho_{\delta}, \varphi \rangle| &= |\langle \mathcal{E}^{\nu}, \varphi - \varphi * \rho_{\delta} \rangle| \\ &\lesssim \left\| E^{\ell, \nu} \right\|_{L^{\frac{p}{2}}_{x,t}} \left(\left\| \partial_{t} (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-2}}_{x,t}} + \left\| u_{\ell}^{\nu} \right\|_{L^{p}_{x,t}} \left\| \nabla (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-3}}_{x,t}} \right) \\ &+ \nu \left\| E^{\ell, \nu} \right\|_{L^{\frac{p}{2}}_{x,t}} \left\| \Delta (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &+ \left\| Q^{\ell, \nu} \right\|_{L^{\frac{p}{2}}_{x,t}} \left\| \nabla (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-3}}_{x,t}} + \left\| C^{\ell, \nu} \right\|_{L^{\frac{p}{3}}_{x,t}} \left\| \varphi - \varphi * \rho_{\delta} \right\|_{L^{\frac{p}{p-3}}_{x,t}} \\ &+ \nu \left(\left\| |\nabla u_{\ell}^{\ell}|^{2} \right\|_{L^{\frac{p}{2}}_{x,t}} + \left\| u^{\nu} - u_{\ell}^{\nu} \right\|_{L^{p}_{x,t}} \left\| \Delta u_{\ell}^{\nu} \right\|_{L^{p}_{x,t}} \right) \left\| \varphi - \varphi * \rho_{\delta} \right\|_{L^{\frac{p}{p-2}}_{x,t}} \\ &+ \nu \left\| u^{\nu} - u_{\ell}^{\nu} \right\|_{L^{p}_{x,t}} \left\| \nabla u_{\ell}^{\nu} \right\|_{L^{p}_{x,t}} \left\| \nabla (\varphi - \varphi * \rho_{\delta}) \right\|_{L^{\frac{p}{p-2}}_{x,t}}. \end{split}$$

Thus, by (2.3), (2.4), (2.7), (2.8) and (2.9) we bound

$$\begin{split} |\langle \mathcal{E}^{\nu} - \mathcal{E}^{\nu} * \rho_{\delta}, \varphi \rangle| &\lesssim \left(\ell^{2\sigma} (\delta + \nu) + \ell^{3\sigma} \delta + \ell^{3\sigma-1} \delta^2 + \nu \ell^{2(\sigma-1)} \delta^2 + \nu \ell^{2\sigma-1} \delta \right) \|\varphi\|_{W^{2,\frac{p}{p-3}}_{x,t}} \\ &\lesssim \left(\ell^{2\sigma} + \ell^{2(\sigma-1)} \delta^2 + \ell^{2\sigma-1} \delta \right) \|\varphi\|_{W^{2,\frac{p}{p-3}}_{x,t}} \lesssim \delta^{2\sigma} \|\varphi\|_{W^{2,\frac{p}{p-3}}_{x,t}} \,, \end{split}$$

where we have used that $\nu < 1$ and in the last inequality we have chosen $\ell = \delta$. Similarly, one can estimate

$$\left| \left\langle \mathcal{E}^{\nu} * \rho_{\delta}, \varphi \right\rangle \right| \lesssim \delta^{2\sigma - 2} \left\| \varphi \right\|_{L^{\frac{p}{p - 3}}_{x,t}}$$

The implicit constants in all the inequalities above depend only on local $L_t^p B_{p,\infty}^{\sigma}$ norms of u^{ν} , which we are assuming to enjoy a bound uniform in viscosity. This proves (4.7). Then, the fact that $\{\mathcal{E}^{\nu}\}_{\nu>0}$ stays bounded in $B_{\frac{p}{2},\infty}^{2(\sigma-1)}$ follows by the same argument of STEP 1 in the proof of Theorem 1.1.

5. DISCUSSION

5.1. Intermittency in turbulence. In this section u^{ν} will denote a sufficiently regular solution to (NS) with $\nu > 0$, in three dimensions. Although the result is true in any spatial dimension $d \ge 2$, the only physically meaningful case is when d = 3. This is because in two dimensions, strong $L^2_{x,t}$ compactness is already inconsistent with a non-trivial dissipation [67]. We introduce the "absolute structure function exponents" ζ_p as

$$\langle |u^{\nu}(x+\ell z)-u^{\nu}(x)|^p\rangle \sim \ell^{\zeta_p} \qquad z \in \mathbb{S}^2, \ p \ge 1.$$

Here the symbol $\langle \cdot \rangle$ denotes some relevant averaging procedure that might be space, time or ensemble. Mathematically, this translates into an exact $B_{p,\infty}^{\sigma_p}$ spatial Besov regularity with $\sigma_p = \frac{\zeta_p}{p}$. Under the assumptions of homogeneity, isotropy, self-similarity and that all the main statistics of the fluid are completely determined by the non-trivial kinetic energy dissipation rate, the Kolmogorov theory of turbulence [65] from 1941 predicts the universal dependence $\zeta_p = \frac{p}{3}$, linear in $p \in [1, \infty]$. This yields to a $\frac{1}{3}$ Hölder exponent of the velocity field uniform in viscosity, space and time, connecting the Kolmogorov statistical framework to the subsequent ideal and deterministic picture of Onsager [77] from 1949. We refer to the recent essay [47] describing the Onsager contributions to the theory of turbulence.

However, there is strong empirical evidence that, in actual turbulence, space-time homogeneity and selfsimilarity break down, making any approach based on the latter inadequate to describe flows at high Reynolds number. This ubiquitous phenomenon is known as "intermittency" [50, Chapter 8]. Despite several results [2,28,49,51,66,71,78,81], a quantitative theoretical understanding of intermittency from first principles is still missing. What appears true from observation is the emergence of a continuous spectrum of Hölder exponents spreading over the domain, possibly resulting into a spotty and fractal distribution of the energetically active regions in space-time [72, 73, 84]. A more precise connection on how to relate anomalies in the exponents ζ_p to a, possibly fractal, lower dimensional dissipation was given by Frisch [49], however, without rigorous mathematical proofs. What our results prove is a quantitative downward deviation from the exponent $\frac{1}{3}$ for all moments p > 3. More precisely, the following is a direct consequence of Corollary 1.2. **Corollary 5.1** (Vanishing viscosity intermittency). On $\mathbb{T}^3 \times (0,T)$, let $\{u^\nu\}_{\nu>0} \subset L^2_t H^1_x$ be a sequence of weak solutions to (NS). Assume that, in the limit as $\nu \to 0$, $\mathcal{E}^\nu := D^\nu + \nu |\nabla u^\nu|^2$ converges in $\mathcal{D}'_{x,t}$ to a non-trivial measure concentrated on a set S with $\dim_{\mathcal{H}} S = \gamma \in [1,4]$. For all $p \in [3,\infty]$ for which there

exists $\zeta_p \in (0,p)$ such that $\{u^{\nu}\}_{\nu>0}$ stays bounded in $L_t^p B_{p,\infty}^{\frac{\varsigma_p}{p}}$, it must hold

$$\zeta_p \le \frac{p}{3} - \frac{2\kappa(p-3)p}{9p-3\kappa(p-3)},\tag{5.1}$$

where $\kappa := 4 - \gamma$ is the codimension of the dissipation concentration set.

Some remarks are in order. The assumption on the distributional convergence of the sequence $\{\mathcal{E}^{\nu}\}_{\nu>0}$ towards a measure is satisfied in several cases, possibly up to subsequences. Indeed, for any sequence of Leray-Hopf weak solutions emanating from L_x^2 bounded initial data, $\{\nu | \nabla u^{\nu} |^2\}_{\nu>0}$ is bounded in $L_{x,t}^1$, and then it admits a weak limit in the sense of measures. Consequently, a subsequence of $\{\mathcal{E}^{\nu}\}_{\nu>0}$ will necessarily converge in $\mathcal{D}'_{x,t}$ to a non-negative measure if $D^{\nu} \equiv 0$, or if $D^{\nu} \geq 0$ and $\{u^{\nu}\}_{\nu>0}$ is compact in $L^3_{x,t}$. Although the latter compactness assumption is mild in terms of observations [18, 42], it is in general not rigorously justified. The restriction to $\gamma \ge 1$, i.e. $\kappa \le 3$, is natural since otherwise finding a $\sigma_p > 0$ for which (1.3) holds would necessarily require p to be quantitatively below $\frac{9}{2}$. This is unnatural in view of the Sobolev embedding $B_{3,\infty}^{\frac{1}{3}} \subset L^{\frac{9}{2}-}$ and the exactness of the Four-Fifths law. In this range of parameters, it is then clear that (5.1) forces the structure functions exponents to quantitatively deviate from the Kolmogorov prediction $\zeta_p = \frac{p}{3}$ for all p > 3 as soon as $\kappa > 0$ and the dissipation is a non-trivial lower dimensional measure. Consequently, intermittency must happen and the larger the p the larger the discrepancy. If $\kappa < 1$, there will be a value $p^* \in (3,\infty)$ at which the right hand side in (5.1) vanishes. In that case, the solution can not have any fractional regularity in L^p for all $p \ge p^*$. The results from [33] can be then applied, matching with the numerology. It is important to note that for p = 3 lower dimensionality does not give any correction, which is consistent with the Kolmogorov Four-Fifths law being exact.

The use of Hausdorff to measure the dimension allows to deduce intermittency even if the concentration set happens to be locally dense. In view of the wild behavior at high Reynolds numbers, this seems to be essential to capture relevant dissipative properties of the flow. Indeed, even simpler dynamical models such as the one-dimensional Burgers equation might exhibit shocks, and thus dissipation, proliferating over a dense set [79]. It is then conceivable to expect an even more complicated scenario in the Navier–Stokes equations, where the already complicated dynamics is enhanced by geometric constraints.

5.2. Comparison with experimental data. The celebrated works of Meneveau and Sreenivasan studied the relationship between properties of the energy dissipation measure to intermittency [72, 73, 84]. These papers suggest from experiments that, in the infinite Reynolds number limit, the anomalous energy dissipation measure at fixed time is concentrated on a fractal subset of dimension less than the space dimension 3, about 2.87, and has volume zero [73]. Moreover, based on the data, it is supposed that this fractal dimension is roughly constant in time, making the inferred space-time support of the dissipation measure to be of dimension 3.87. We now give an argument for this value based on the formalism developed in this paper. It seems reasonable that our considerations are sharp locally⁴ at p = 3. Set

$$\zeta_p^* := \frac{p}{3} - \frac{2\kappa(p-3)p}{9p - 3\kappa(p-3)},$$

i.e. the right hand side in (5.1). In dimension d = 3, it is readily verified that

$$\left. \frac{d\zeta_p^*}{dp} \right|_{p=3} = \frac{3-2\kappa}{9} = \frac{2\gamma-5}{9}$$

High resolution direct numerical simulations of incompressible turbulence [63] indicate $\frac{d\zeta_p}{dp}\Big|_{p=3} \approx 0.3$. This corresponds to $\gamma = 3.85$, remarkably close to the observations of Meneveau and Sreenivasan. See Figure 1 for an inspection of numerical data and this bound.

⁴We are mainly appealing to two facts for $p \approx 3$: the dissipation is what constraints the regularity the most and the saturation of our upper bounds. Modulo the possible gap of longitudinal vs. absolute increments, the exactness of the Four-Fifths law makes both claims valid at p = 3.



FIGURE 1. Structure function exponents for $p \in [3, 6]$. Blue dots are absolute structure function exponents measured from the JHU turbulence database. Red triangles are transverse exponents reported in [63]. Dashed grey line corresponds to the Kolmogorov prediction of $\frac{p}{3}$. Solid grey line corresponds to our bound ζ_p^* with $\gamma = 3.85$ inferred from [63].

5.3. Sharpness of the results & convex integration. Recent years have seen a quite intense mathematical work [12, 13, 30, 32, 61] in producing Hölder continuous Euler flows with non-constant kinetic energy, culminating in the resolution of the Onsager conjecture [14, 57]. See also [29, 36, 60] for further developments. Hölder continuous weak solutions with non-negative D have also been produced [31, 58], getting closer to the physical case in which Euler arises in the inviscid limit of Navier–Stokes. The method goes under the name of "convex integration", introduced in this context by De Lellis and Székelyhidi [32]. The Onsager conjecture being true validates the Kolmogorov self-similar and isotropic prediction of a spatially homogeneous $\frac{1}{3}$ Hölder exponent, although only in the ideal inviscid setting. Very recent works [10, 11, 82] also prove anomalous dissipation in the inviscid limit, but with the use of an external forcing term. Remarkably, the convex integration methods have been recently modified to incorporate intermittency [15, 53, 54, 75], producing Euler weak solutions with a non-trivial dependence of ζ_p on p, thus going beyond Hölder regularity.

Producing weak solutions of (E) with a non-trivial lower dimensional non-negative measure D is not yet done. However, the very recent intermittent constructions [53, 54] are going into a direction that might tackle the sharpness of Theorem 1.1. There are reasons to believe that the results of this paper are sharp. What is certainly true is that the negative Besov regularity $D \in B_{\frac{p}{3},\infty}^{\frac{2\sigma}{1-\sigma}-1}$ can not be improved. Indeed, in [36, 61] Hölder continuous weak solutions with a prescribed kinetic energy with time Hölder regularity arbitrarily close to $\frac{2\sigma}{1-\sigma}$ have been constructed. Since the derivative of the kinetic energy coincides with the time marginal of D, by choosing a non-constant Cantor function whose derivative concentrates on a given set with Hausdorff dimension arbitrarily close to $\frac{2\sigma}{1-\sigma}$, we deduce that in this case D can not posses any fractional regularity larger than $\frac{2\sigma}{1-\sigma} - 1$, al least when $p = \infty$. Note that for the solutions constructed in [36, 61] D is only a space-time distribution, not necessarily a measure. With that being said, it would be nice to know how sharp this is in space-time and for all values of $p \geq 3$, perhaps showing full flexibility for Euler flows in saturating any geometrical/analytical constraint imposed by the PDE. The work [34] is also closely related to this discussion, although imposing the additional constraint on the solution being smooth outside the (closed) dissipative set of times.

An interesting feature of the regularity $D \in B_{\frac{1}{2},\infty}^{\frac{2\sigma}{1},\sigma-1}$ is that it does not seem to be possible to achieve by showing that an approximation of D stays bounded in that space. For instance, although a natural candidate would be the viscous dissipation $\mathcal{E}^{\nu} = D^{\nu} + \nu |\nabla u^{\nu}|^2$, it does not seem plausible to improve the uniform bound of Corollary 4.13. The same seems to happen with both the Duchon–Robert approximation [43] and the Constantin–E–Titi one [24], somehow suggesting that all the available approximations do not capture essential cancellations which however appear in the limiting object. In particular, fine dissipative mechanism arising in the limit $\nu \to 0$ might stand apart from their measurements at very high, but finite, Reynolds numbers.

On a different side than constructing solutions, there are some other recent works addressing the issue of intermittency. For instance, an extensive "volumetric approach" has been developed in [19, 20] to extract

information from the most energetically active parts of the flow at a given scale. This allows to analytically define a notion of dimension and, among other things, to validate the Frisch–Parisi multifractal formalism [51]. See also [5,6,64]. More related to the spirit of this paper are [33,35]. In particular, in [35] intermittency for Besov solutions is deduced assuming lower dimensionality of the dissipation in the Minkowski sense, while in [33] by means of the Hausdorff dimension but only for integrable weak solutions, thus failing in making any non-trivial use of fractional regularity. The current paper closes the gap and finally reconciles the two approaches.

5.4. General open sets. Although all the results in this paper are stated in the spatially periodic setting \mathbb{T}^d , we emphasize that they are all intrinsically local. Thus, they carry over any open set $\Omega \subset \mathbb{R}^d$, of course away from the boundary. To do that rigorously, the only thing that has to be fixed is the usual issue with the pressure. The latter being determined only up to arbitrary time dependent functions might be in general not enough to deduce the double pressure regularity, even if only in the interior. As shown in the lemma below, it is enough that the spatial average of |q| has a suitable time integrability.

Lemma 5.2. Let $\Omega \subset \mathbb{R}^d$ be open. Assume

$$-\Delta q = \operatorname{div}\operatorname{div}(u \otimes u) \qquad \text{in } \Omega \times (0,T).$$

Let $p \in (2,\infty]$ and $\sigma \in (0,\frac{1}{2})$. If $q \in L_t^{\frac{p}{2}}L_x^1$ and $u \in L_t^p B_{p,\infty}^{\sigma}$ locally, then $q \in L_t^{\frac{p}{2}}B_{\frac{p}{2},\infty}^{2\sigma}$ locally

Proof. Let $U \subset \Omega$ be any open set. In what follows t is any fixed instant of time, picked in a full measure subset of (0,T). Since $u(t) \in B_{p,\infty}^{\sigma}$ locally inside Ω , we find a divergence-free and compactly supported $\tilde{u}(t) \in B_{p,\infty}^{\sigma}$ on \mathbb{R}^d such that $\tilde{u}(t) \equiv u(t)$ on U. Let \tilde{q} be the unique solution to

 $-\Delta \tilde{q} = \operatorname{div} \operatorname{div}(\tilde{u} \otimes \tilde{u}) \qquad \text{in } \mathbb{R}^d$

decaying at infinity. By [22,23,59] and the continuity of the extension operator

$$\|\tilde{q}(t)\|_{B^{2\sigma}_{\frac{p}{2},\infty}} \lesssim \|\tilde{u}(t)\|^2_{B^{\sigma}_{p,\infty}} \lesssim \|u(t)\|^2_{B^{\sigma}_{p,\infty}(U)},$$

from which we deduce $\tilde{q} \in L_t^{\frac{p}{2}} B_{\frac{p}{2},\infty}^{2\sigma}$. Since $q = q - \tilde{q} + \tilde{q}$, we are only left to prove $q - \tilde{q} \in L_t^{\frac{p}{2}} B_{\frac{p}{2},\infty}^{2\sigma}$ locally inside $U \times (0,T)$. However, since $(q - \tilde{q})(t)$ is harmonic in U, the mean value property implies that its C_x^1 norm on compact subsets of U is bounded by the L_x^1 norm on U. Since $q \in L_t^{\frac{p}{2}} L_x^1$ by assumption, the proof is concluded.

Once double pressure regularity holds, all the results in this paper can be replicated in the interior of any open set Ω .

5.5. Intermittency in scalar turbulence. Let $\Omega \subset \mathbb{R}^d$ be open. Given an incompressible vector field $v: \Omega \times (0,T) \to \mathbb{R}^d$, consider the transport equation

$$\partial_t \theta + \operatorname{div}(\theta v) = 0 \qquad \text{in } \Omega \times (0, T).$$
 (T)

The local dissipation \tilde{D} can be defined as

$$\partial_t \frac{|\theta|^2}{2} + \operatorname{div}\left(\frac{|\theta|^2}{2}v\right) = -\tilde{D} \quad \text{in } \mathcal{D}'_{x,t}$$

as soon as $v \in L_{x,t}^p$, $\theta \in L_{x,t}^s$ with $\frac{1}{p} + \frac{2}{s} \leq 1$. Note that, being a linear equation, weak solutions can be obtained from weak compactness of vanishing diffusivity approximations. For such solutions, it follows that \tilde{D} is actually a non-negative Radon measure.

Passive scalar transport is a well studied physical system, and there is a wealth of evidence for anomalous dissipation therein [37,83]. Obukhov [76] and Corrsin [27] derived Onsager-type predictions on the requisite degree of singularity required to see anomalous dissipation in this context. The result is, roughly, that if σ represents the fractional regularity of the velocity, the scalar cannot have regularity β greater than $\frac{1-\sigma}{2}$. This was made rigorous by Eyink [45], following the works of Constantin and Procaccia [25, 26]. See also the discussion in [39]. Recently, there have been mathematical constructions of passive scalars exhibiting

anomalous dissipation [3, 16, 39, 55], even some that nearly exhibit the sharpness of the Obukhov–Corrsin theory in Hölder spaces [21, 44]. The end point cases remain open. In physical situations, however, it is widely expected that the scalar is far from monofractal, and displays anomalous scaling exponents, e.g. the Obukhov–Corrsin theory in Hölder spaces is far from sharp and instead it holds only in an appropriate critical class: L_x^2 on the scalar rather than L_x^{∞} if the velocity is Hölder continuous. This expectation has a great deal of numerical justification [62, 83] and a theoretical one in the Kraichnan model of passive scalar turbulence [8, 52].

All of the analysis we preformed for the Euler equations can be carried over to the setting of passive scalars and gives precise constraints on the relation between intermittency and dissipation measure. In the following theorem we highlight the main implications of our framework, as it applies to this situation.

Theorem 5.3. Let $v \in L_{x,t}^p$ be a given vector field for some $p \in [1,\infty]$. Let $\theta \in L_{x,t}^s$ be a weak solution to (T) with $\frac{1}{p} + \frac{2}{s} \leq 1$ and with local dissipation $\tilde{D} \in \mathcal{D}'_{x,t}$. Set

$$\begin{split} \tilde{E}^{\ell} &:= \frac{\left|\theta - \theta_{\ell}\right|^{2}}{2}, \qquad \tilde{Q}^{\ell} &:= \frac{\left|\theta - \theta_{\ell}\right|^{2}}{2}(v - v_{\ell}), \qquad \tilde{R}^{\ell} &:= \theta_{\ell}v_{\ell} - (\theta v)_{\ell}\\ and \qquad \tilde{C}^{\ell} &:= (\theta - \theta_{\ell})\left((v - v_{\ell}) \cdot \nabla \theta_{\ell} + \operatorname{div} \tilde{R}^{\ell}\right), \end{split}$$

where the subscripts ℓ denote the space mollification. For all $\ell > 0$, the following identity holds

$$-\tilde{D} = (\partial_t + v_\ell \cdot \nabla)\tilde{E}^\ell + \operatorname{div} \tilde{Q}^\ell + \tilde{C}^\ell \qquad in \ \mathcal{D}'_{x,t}.$$

Assume that $v \in L^p_t B^{\sigma}_{p,\infty}$ and $\theta \in L^s_t B^{\beta}_{s,\infty}$ for some $\sigma, \beta \in (0,1)$. Then $\tilde{D} \in B^{\frac{2\beta}{1-\sigma}-1}_{\frac{ps}{2p+s},\infty}$ locally. If in addition \tilde{D} is a real-valued Radon measure, we also have $|\tilde{D}| \ll \mathcal{H}^{\gamma}$ for any $\gamma \geq 0$ such that

$$\frac{2\beta}{1-\sigma} > 1 - \frac{p(s-2)-s}{ps}(d+1-\gamma).$$

Note that $|\theta|^2 v \in L_{x,t}^{\frac{ps}{2p+s}}$ and its Hölder dual is precisely $\frac{ps}{p(s-2)-s}$, making clear their appearance in the theorem above. An immediate implication of Theorem 5.3 is the following intermittency-type statement. Since the arguments are the same we have given for Euler, we will omit the proof.

Corollary 5.4. Let $v \in L_{x,t}^p$ be a given vector field for some $p \in [1, \infty]$. Let $\theta \in L_{x,t}^s$ be a weak solution to (\mathbf{T}) with $\frac{1}{p} + \frac{2}{s} \leq 1$ and with a non-trivial local dissipation measure \tilde{D} concentrated on a space-time set S with $\dim_{\mathcal{H}} S = \gamma$. For all such p and s for which there exist $\sigma_p, \beta_s \in (0, 1)$ such that $v \in L_t^p B_{p,\infty}^{\sigma_p}$ and $\theta \in L_t^s B_{s,\infty}^{\beta_s}$, it must hold

$$\frac{2\beta_s}{1-\sigma_p} \le 1 - \frac{p(s-2)-s}{ps}(d+1-\gamma).$$

This results immediately recovers the Obukhov–Corrsin and Eyink bounds, which say that non-trivial dissipation requires $\beta_s \leq \frac{1-\sigma_p}{2}$ for all $p \in [1, \infty]$, and represents a substantial refinement in case more is known about the dissipation measure. For instance, numerical work on scalar advection by three dimensional turbulent velocity fields [62] suggests that exponents saturate $s\beta_s \to 1.2$ as $s \to \infty$, approximately, thus quite far from being monofractal. We believe that constructions in the spirit of [3, 16, 21, 39, 44, 55] could be made to show the sharpness of this intermittent Obukhov–Corrsin theory.

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