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Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations

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Abstract. We study the local equation of energy for weak solutions of three-dimensional incompressible Navier–Stokes and Euler equations. We define a dissipation term $D(\boldsymbol{u})$ which stems from an eventual lack of smoothness in the solution $\boldsymbol{u}$. We give in passing a simple proof of Onsager's conjecture on energy conservation for the three-dimensional Euler equation, slightly weakening the assumption of Constantin *et al*. We suggest calling weak solutions with non-negative $D(\boldsymbol{u})$ 'dissipative'.

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1. Introduction

Here we consider the three-dimensional (for the most part) incompressible Navier–Stokes and Euler equations. For simplicity we limit ourselves to flows on the torus $\mathcal{T} = (\mathbb{R}/\mathbb{Z})^3$, i.e. with periodic boundary conditions.

Let us take the Navier–Stokes equation first. For an initial velocity field $\boldsymbol{u}_0$ with finite energy, as is well known (Leray [4, 5]), there exists at least one weak solution (i.e. in the sense of distributions) to the Cauchy problem. *A priori* such a solution belongs to $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ and there is not enough smoothness to ensure the classical energy equality; all we know is that one can define some weak solution satisfying, in addition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}\boldsymbol{x} + \nu\int(\nabla\boldsymbol{u})^2\,\mathrm{d}\boldsymbol{x} \leqslant 0.$$

As a first step we show that for any weak solution $\boldsymbol{u}$ of the Navier–Stokes equation, the local equation of energy

$$\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) + \mathrm{div}(\boldsymbol{u}(\tfrac{1}{2}\boldsymbol{u}^2 + p)) - \nu\Delta\tfrac{1}{2}\boldsymbol{u}^2 + \nu(\nabla\boldsymbol{u})^2 + D(\boldsymbol{u}) = \boldsymbol{0}$$

is satisfied, with $D(\boldsymbol{u})$ defined in terms of the local smoothness of $\boldsymbol{u}$. Thus the non-conservation of energy originates from two sources: viscous dissipation and a possible lack of smoothness in the solution.

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For the Euler equation, we consider weak solutions in $L^3(0, T; L^3)$. Although there is no general result at present for the global-in-time existence of such solutions, some examples are known (consider any two-dimensional weak solution given by Yudovich's [10] theorem).

According to an approach in the study of turbulence that goes back to Onsager [7], it might be true that such weak solutions of the three-dimensional Euler equation describe the turbulent flow correctly (in the limit of infinite Reynolds number of course). Smooth solutions conserve energy as is shown by a simple integration by parts, but this calculation does not extend to weak solutions. Some weak solutions have been constructed without energy conservation (Scheffer [9], Shnirelman [8]). Onsager had conjectured that weak solutions of the Euler equation satisfying a Hölder continuity condition of order $>\frac{1}{3}$ should conserve energy. The great interest of this question was duly emphasized by Eyink [2], who also gave a proof of energy conservation under a stronger assumption. Then Constantin *et al* [1] gave a simple and elegant proof of energy conservation under the weaker and more natural assumption that $\boldsymbol{u}$ belongs to the Besov space $B_3^{\alpha,\infty}$ with $\alpha > \frac{1}{3}$.

Our considerations above on dissipation in the Navier–Stokes equation apply to weak solutions of Euler as well: one has a local equation of energy

$$\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) + \operatorname{div}(\boldsymbol{u}(\tfrac{1}{2}\boldsymbol{u}^2 + p)) + D(\boldsymbol{u}) = \boldsymbol{0}$$

and the explicit form of $D(\boldsymbol{u})$ makes it possible to prove energy conservation under a slightly weaker assumption.

We then come to the problem of distinguishing, among weak solutions of Euler or Navier–Stokes equations, which ones may be considered physically acceptable. We first see that the weak solutions of Navier–Stokes constructed by Leray [4, 5] do satisfy $D(\boldsymbol{u}) \geqslant 0$. We also show that any weak solution of the Euler equation which is a strong limit of smooth solutions of the Navier–Stokes equation satisfies this same condition. Finally, we are led to a definition of dissipative weak solutions: those satisfying $D(\boldsymbol{u}) \geqslant 0$.

2. The local equation of energy for weak solutions of Navier–Stokes and Euler equations

Our main point is expressed in the following two results:

Proposition 1. *Let $\boldsymbol{u} \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2)$, a weak solution of the Navier–Stokes equation on the three-dimensional torus $\mathcal{T}$:*

$$\begin{aligned} &\partial_t\boldsymbol{u} + \partial_i(u_i\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla p = 0 \\ &\operatorname{div}\boldsymbol{u} = 0. \end{aligned} \tag{1}$$

Let φ be any infinitely differentiable function with compact support on $\mathbb{R}^3$, even, non-negative with integral 1 and $\varphi^\varepsilon(\xi) = (1/\varepsilon^3)\varphi(\xi/\varepsilon)$.

Put $D_\varepsilon(\boldsymbol{u})(x) = \frac{1}{4}\int \nabla\varphi^\varepsilon(\xi)\cdot\delta\boldsymbol{u}(\delta\boldsymbol{u})^2\,\mathrm{d}\xi$, where $\delta\boldsymbol{u} = \boldsymbol{u}(x+\xi) - \boldsymbol{u}(x)$.

Then, as ε goes to 0, the functions $D_\varepsilon(\boldsymbol{u})$ (which are in $L^1(]0, T[\times\mathcal{T})$) converge, in the sense of distributions on $]0, T[\times\mathcal{T}$, towards a distribution $D(\boldsymbol{u})$, not depending on φ, and the following local equation of energy is satisfied:

$$\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) + \operatorname{div}(\boldsymbol{u}(\tfrac{1}{2}\boldsymbol{u}^2 + p)) - \nu\Delta\tfrac{1}{2}\boldsymbol{u}^2 + \nu(\nabla\boldsymbol{u})^2 + D(\boldsymbol{u}) = 0.$$

Proof. Using Sobolev inclusion of H^1 in L^6, one easily sees that $\boldsymbol{u}$ is in $L^3(0, T; L^3)$ and therefore u_iu_k is in $L^{3/2}(0, T; L^{3/2})$; and the same for p since, taking the divergence of (1), one obtains

$$-\Delta p = \partial_k\partial_i(u_iu_k)$$

and if p is the only solution with mean zero, the linear operator $u_i u_k \to p$ is strongly continuous on L^q for $1 < q < \infty$, and so $p \in L^{3/2}(0,T;L^{3/2})$.

Now let us regularize equation (1): denoting $\boldsymbol{u}^\varepsilon = \varphi^\varepsilon * \boldsymbol{u}$, $p^\varepsilon = \varphi^\varepsilon * p$, $(u_i\boldsymbol{u})^\varepsilon = \varphi^\varepsilon * (u_i\boldsymbol{u})$, etc one has

$$\partial_t \boldsymbol{u}^\varepsilon + \partial_i (u_i\boldsymbol{u})^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = 0.$$

This equation multiplied scalarly by $\boldsymbol{u}$, plus equation (1) multiplied by $\boldsymbol{u}^\varepsilon$, gives

$$\partial_t(\boldsymbol{u}\cdot\boldsymbol{u}^\varepsilon) + \operatorname{div}((\boldsymbol{u}\cdot\boldsymbol{u}^\varepsilon)\boldsymbol{u} + p^\varepsilon\boldsymbol{u} + p\boldsymbol{u}^\varepsilon) + E_\varepsilon - \nu\Delta(\boldsymbol{u}\cdot\boldsymbol{u}^\varepsilon) + 2\nu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\varepsilon = 0$$

where

$$E_\varepsilon(t,x) = \partial_i(u_iu_j)^\varepsilon u_j - u_iu_j\partial_i u_j^\varepsilon.$$

Since $\boldsymbol{u} \in L^3(0,T;L^3)$, $\boldsymbol{u}\cdot\boldsymbol{u}^\varepsilon$ converges to $\boldsymbol{u}^2$ and $(\boldsymbol{u}\cdot\boldsymbol{u}^\varepsilon)\boldsymbol{u} + p^\varepsilon\boldsymbol{u} + p\boldsymbol{u}^\varepsilon$ converges to $(\boldsymbol{u}^2 + 2p)\boldsymbol{u}$ in the sense of distributions on $]0,T[\times\mathcal{T}$. Moreover, $\nabla\boldsymbol{u}^\varepsilon$ tends to $\nabla\boldsymbol{u}$ strongly in $L^2(]0,T[\times\mathcal{T})$, thus $E_\varepsilon(t,x)$ converges in the sense of distributions towards

$$-\partial_t(\boldsymbol{u}^2) - \operatorname{div}(\boldsymbol{u}(\boldsymbol{u}^2 + 2p)) + \nu\Delta\boldsymbol{u}^2 - 2\nu(\nabla\boldsymbol{u})^2.$$

Another calculation gives

$$\int \nabla\varphi^\varepsilon(\xi)\cdot\delta\boldsymbol{u}(\delta\boldsymbol{u})^2\,\mathrm{d}\xi = -\partial_i(u_iu_ju_j)^\varepsilon + 2\partial_i(u_iu_j)^\varepsilon u_j + \partial_i(u_ju_j)^\varepsilon u_i - 2u_iu_j\partial_iu_j^\varepsilon.$$

However, $\partial_i(u_ju_j)^\varepsilon u_i = \partial_i(u_i(u_ju_j)^\varepsilon)$, due to the incompressibility of $\boldsymbol{u}$.

Moreover, $\partial_i(u_i(u_ju_j)^\varepsilon - (u_iu_ju_j)^\varepsilon)$ tends to 0 in the sense of distributions on $]0,T[\times\mathcal{T}$ and thus $\int\nabla\varphi^\varepsilon(\xi)\cdot\delta\boldsymbol{u}(\delta\boldsymbol{u})^2\,\mathrm{d}\xi$ has the same limit as $2E_\varepsilon$. □

The same reasoning applies entirely for a weak solution of the Euler equation ($\nu = 0$) and gives

Proposition 2. *Let $\boldsymbol{u} \in L^3(0,T;L^3)$ be a weak solution of the Euler equation. Then the functions $D_\varepsilon(\boldsymbol{u})$ converge, in the sense of distributions, to a distribution $D(\boldsymbol{u})$, not depending on φ, and the following local equation of energy holds:*

$$\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) + \operatorname{div}(\boldsymbol{u}(\tfrac{1}{2}\boldsymbol{u}^2 + p)) + D(\boldsymbol{u}) = 0.$$

Remark. In the two previous propositions $D(\boldsymbol{u})$ measures a possible dissipation (or production) of energy caused by a lack of smoothness in the velocity field $\boldsymbol{u}$, this term is by no means related to the presence or absence of viscosity.

Now let us state a simple smoothness condition which implies $D(\boldsymbol{u}) = 0$.

Proposition 3. *Let $\boldsymbol{u}$ satisfy $\int|\boldsymbol{u}(t,x+\xi) - \boldsymbol{u}(t,x)|^3\,\mathrm{d}x \leqslant C(t)|\xi|\sigma(|\xi|)$, where $\sigma(a)$ tends to 0 with a, and $\int_0^T C(t)\,\mathrm{d}t < +\infty$. Then $D(\boldsymbol{u}) = 0$.*

Proof. One has

$$\left|\int\nabla\varphi^\varepsilon(\xi)\cdot\delta\boldsymbol{u}(\delta\boldsymbol{u})^2\,\mathrm{d}\xi\right| \leqslant \int\left|\nabla\varphi^\varepsilon(\xi)\right||\delta\boldsymbol{u}|^3\,\mathrm{d}\xi$$

integrating over $]0,T[\times\mathcal{T}$ yields

$$\int \mathrm{d}t \int \left|D_\varepsilon(\boldsymbol{u})\right| \mathrm{d}x \leqslant \int \mathrm{d}t \int \left|\nabla\varphi^\varepsilon(\xi)\right| \mathrm{d}\xi \int \left|\delta\boldsymbol{u}\right|^3 \mathrm{d}x$$
$$\leqslant \int_0^T C(t)\,\mathrm{d}t \int \frac{1}{\varepsilon^4}\left|\nabla\varphi\left(\frac{\xi}{\varepsilon}\right)\right| |\xi|\sigma(|\xi|)\,\mathrm{d}\xi$$

and putting $\xi = \varepsilon\eta$, one can see that this tends to 0 with ε. □

Remark. If $\boldsymbol{u}$ is a weak solution of the Euler equation and satisfies the smoothness condition in proposition 2 above, then the kinetic energy of $\boldsymbol{u}$ is conserved (just integrate the local equation of energy over x). This provides a proof of Onsager's conjecture [1, 2, 7] under an assumption slightly weaker than $\boldsymbol{u} \in L^3(0,T;B_3^{\alpha,\infty})$ with $\alpha > \frac{1}{3}$.

3. Relevance to real turbulence?

There is still some doubt as to whether weak solutions of the Navier–Stokes equation, the uniqueness of which is unknown, or hypothetical weak solutions of the Euler equation, are relevant to the description of turbulent flows at high Reynolds number. It seems reasonable to require some extra conditions: one of them might be that the lack of smoothness could not lead to local energy creation. In other words, one should have $D(\boldsymbol{u}) \geqslant 0$ on $]0,T[\times\mathcal{T}$.

It is quite remarkable that this condition is satisfied by every weak solution of the Navier–Stokes equation obtained as a limit of (a subsequence of) solutions $\boldsymbol{u}_\varepsilon$ of the regularized equation introduced by Leray [4, 5]:

$$\partial_t\boldsymbol{u}_\varepsilon + \partial_i((\varphi^\varepsilon * \boldsymbol{u}_{\varepsilon i})\boldsymbol{u}_\varepsilon) - \nu\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = 0$$
$$\mathrm{div}(\boldsymbol{u}_\varepsilon) = 0 \qquad \boldsymbol{u}_\varepsilon(0) = \varphi^\varepsilon * \boldsymbol{u}_0.$$

For $\boldsymbol{u}_0$ given in L^2 and $\varepsilon > 0$, this equation has a unique C^∞ solution $\boldsymbol{u}_\varepsilon$.

The sequence $(\boldsymbol{u}_\varepsilon)$ is bounded in $L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ and a subsequence converges to $\boldsymbol{u}$, a weak solution of Navier–Stokes, weakly in $L^2(0,T;H^1)$ and strongly in $L^3(0,T;L^3)$. However, for the regularized equation, one has the local energy balance

$$\partial_t\left(\tfrac{1}{2}\boldsymbol{u}_\varepsilon^2\right) + \mathrm{div}\left(\left(\varphi^\varepsilon * \boldsymbol{u}_\varepsilon\right)\tfrac{1}{2}\boldsymbol{u}_\varepsilon^2 + p_\varepsilon\boldsymbol{u}_\varepsilon\right) - \nu\Delta\tfrac{1}{2}\boldsymbol{u}_\varepsilon^2 + \nu\left(\nabla\boldsymbol{u}_\varepsilon\right)^2 = 0$$

hence $\nu(\nabla\boldsymbol{u}_\varepsilon)^2$ converges in the sense of distributions towards

$$-\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) - \mathrm{div}(\boldsymbol{u}(\tfrac{1}{2}\boldsymbol{u}^2 + p)) + \nu\Delta\tfrac{1}{2}\boldsymbol{u}^2.$$

For every function $\psi(t,x)$ infinitely differentiable and non-negative, the functional $\boldsymbol{u} \to \iint(\nabla\boldsymbol{u})^2\,\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t$ is convex and lower semicontinuous on the weak space $L^2(0,T;H^1)$, and thus

$$\lim_{\varepsilon\to 0}\iint(\nabla\boldsymbol{u}_\varepsilon)^2\,\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint(\nabla\boldsymbol{u})^2\,\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

which implies $\lim_{\varepsilon\to 0}\nu(\nabla\boldsymbol{u}_\varepsilon)^2 - \nu(\nabla\boldsymbol{u})^2 = D(\boldsymbol{u}) \geqslant 0$. This fact is well known; see, for example, [6].

Remark. Two natural questions arise at this point:

(a) Does there exist a weak solution of Navier–Stokes in $L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ with $D(\boldsymbol{u}) \neq 0$?

(b) Does the condition $D(\boldsymbol{u}) \geqslant 0$ imply uniqueness for weak solutions of Navier–Stokes?

Let us call 'dissipative' such weak solutions with $D(\boldsymbol{u}) \geqslant 0$.

In the case of the inviscid Burgers equation in one space dimension, $D(\boldsymbol{u}) \geqslant 0$ coincides with the usual entropy condition of negative jumps, which does imply uniqueness.

The following proposition shows that the condition $D(\boldsymbol{u}) \geqslant 0$ appears naturally for weak solutions of the Euler equation.

Proposition 4. *Let $\boldsymbol{u} \in L^3(0, T; L^3)$ be a weak solution of the Euler equation, which is the strong limit of a sequence of dissipative weak solutions of Navier–Stokes as viscosity goes to zero. Then $D(\boldsymbol{u}) \geqslant 0$.*

Proof. The weak solution of Navier–Stokes $\boldsymbol{u}^\nu$ satisfies

$$\partial_t(\tfrac{1}{2}\boldsymbol{u}^{\nu 2}) + \operatorname{div}((\tfrac{1}{2}\boldsymbol{u}^{\nu 2} + p^\nu)\boldsymbol{u}^\nu) - \nu\Delta\tfrac{1}{2}\boldsymbol{u}^{\nu 2} + \nu(\nabla\boldsymbol{u}^\nu)^2 + D(\boldsymbol{u}^\nu) = 0.$$

Since $\boldsymbol{u}^\nu$ tends to $\boldsymbol{u}$ in $L^3(0, T; L^3)$ strong, one has

$$\lim_{\nu\to 0}(\nu(\nabla\boldsymbol{u}^\nu)^2 + D(\boldsymbol{u}^\nu)) = -\partial_t(\tfrac{1}{2}\boldsymbol{u}^2) - \operatorname{div}((\tfrac{1}{2}\boldsymbol{u}^2 + p)\boldsymbol{u}) = D(\boldsymbol{u})$$

in the sense of distributions, and thus $D(\boldsymbol{u}) \geqslant 0$. □

Remark. Let $\boldsymbol{u} \in L^3(0, T; L^3)$ be a weak solution of the Euler equation, dissipative in the sense that $D(\boldsymbol{u}) \geqslant 0$. Then it is a dissipative solution of the Euler equation in the sense of Lions [6]. Indeed, every weak solution with $(\mathrm{d}/\mathrm{d}t)\int \frac{1}{2}\boldsymbol{u}^2\,\mathrm{d}x \leqslant 0$ is a dissipative solution in Lions' sense. Notice that this last condition does not prevent *a priori* local creation of energy in some regions of the flow.

4. The two-dimensional case

In two space dimensions the situation is clearer for the Navier–Stokes equation. For every initial velocity field $\boldsymbol{u}_0$ in L^2 one has a unique weak solution in $L^2(0, T; H^1) \cap L^\infty(0, T; L^2)$ and this solution satisfies the global energy balance

$$\frac{1}{2}\int \boldsymbol{u}^2(T, x)\,\mathrm{d}x + \nu\int_0^T \mathrm{d}t\int(\nabla\boldsymbol{u})^2\,\mathrm{d}x = \frac{1}{2}\int \boldsymbol{u}_0^2(x)\,\mathrm{d}x.$$

In fact, one has a slightly stronger result:

Proposition 5. *Let $\boldsymbol{u}$ be the unique weak solution of the two-dimensional Navier–Stokes equation above. Then $D(\boldsymbol{u}) = 0$.*

Proof. We use the interpolation inequality $\|\boldsymbol{v}\|_{L^3} \leqslant C\|\boldsymbol{v}\|_{L^2}^{2/3}\|\boldsymbol{v}\|_{H^1}^{1/3}$ together with $\|\delta\boldsymbol{u}\|_{L^2} \leqslant |\xi|\|\boldsymbol{u}\|_{H^1}$ and $\|\delta\boldsymbol{u}\| \leqslant 2\|\boldsymbol{u}\|$ (for any norm).

From the expression for $D_\varepsilon(\boldsymbol{u})$ one has

$$\|D_\varepsilon(\boldsymbol{u})\|_{L^1(\mathrm{d}x)} \leqslant \frac{1}{4\varepsilon}\iint |\nabla\varphi(\xi)||\boldsymbol{u}(t, x + \varepsilon\xi) - \boldsymbol{u}(t, x)|^3\,\mathrm{d}x\,\mathrm{d}\xi$$

and since $\boldsymbol{u} \in L^\infty(0, T; L^2)$, this is bounded from above, for almost every $t \in [0, T]$, by a fixed integrable function $C\|\boldsymbol{u}(t)\|_{H^1}^2$.

On the other hand, for almost every $t \in [0, T]$, $\boldsymbol{u}(t)$ is in H^1 and

$$\left\|\boldsymbol{u}(t, x + \varepsilon\xi) - \boldsymbol{u}(t, x)\right\|_{L^3(\mathrm{d}x)}^3 \leqslant C\varepsilon^2|\xi|^2\|u\|_{H^1}^3$$

so that $\|D_\varepsilon(\boldsymbol{u})\|_{L^1(\mathrm{d}x)} \to 0$ as ε goes to 0.

Applying Lebesgue's dominated convergence theorem, one obtains $D(\boldsymbol{u}) = 0$. □

The case of the 2D Euler equation

For $\boldsymbol{u}_0$ in L^2 such that $\omega_0 = \operatorname{curl} \boldsymbol{u}_0 \in L^r$, $1 < r < \infty$, there exists at least one weak solution of the Euler equation in the space $C([0, \infty[; W^{1,r})$ [6]. From Sobolev inclusion, for $r \geqslant \frac{6}{5}$ one has $W^{1,r} \subset L^3$, then $D(\boldsymbol{u})$ is defined and the local energy balance holds with $D(\boldsymbol{u})$.

Moreover, we have

Proposition 6. *Let $\boldsymbol{u}$ be a weak solution of the 2D Euler equation as above with $r > \frac{3}{2}$, then* $D(\boldsymbol{u}) = 0$.

Proof. Apply the Hölder inequality

$$\|\delta\boldsymbol{u}\|_{L^3} \leqslant \|\delta\boldsymbol{u}\|_{L^r}^{\alpha} \|\delta\boldsymbol{u}\|_{L^q}^{1-\alpha} \qquad \frac{1}{3} = \frac{\alpha}{r} + \frac{1-\alpha}{q}.$$

Taking $q = 2r/(2-r)$, so that $\| \ \|_{L^q} \leqslant c\| \ \|_{W^{1,r}}$, and using $\|\delta\boldsymbol{u}\|_{L^r} \leqslant |\xi| \|\boldsymbol{u}\|_{W^{1,r}}$, one obtains

$$\|\delta\boldsymbol{u}\|_{L^3} \leqslant c|\xi|^{\alpha} \|\boldsymbol{u}\|_{W^{1,r}} \qquad \text{with} \quad \alpha = \frac{5}{3} - \frac{2}{r}.$$

If $r > \frac{3}{2}$, then $\alpha > \frac{1}{3}$ and proposition 3 applies. □

5. Inertial dissipation and the four-fifth law

We have already seen that $D(\boldsymbol{u})$ does not depend on φ. Assuming some space continuity of $\boldsymbol{u}$, we are able to express it more explicitly using a radially symmetric function $\varphi(|\xi|)$.

Let us put

$$S(\boldsymbol{u})(x, r) = \int_{|\xi|=1} (\boldsymbol{u}(x + r\xi) - \boldsymbol{u}(x))^2 (\boldsymbol{u}(x + r\xi) - \boldsymbol{u}(x)) \cdot \xi \, \mathrm{d}\Sigma(\xi)$$

where $\mathrm{d}\Sigma$ denotes the area measure on the sphere.

An easy computation gives

$$D_\varepsilon(\boldsymbol{u}) = \tfrac{1}{4} \int_0^\infty \varphi'(r)\, r^3 \frac{S(\boldsymbol{u})(x, \varepsilon r)}{\varepsilon r} \, \mathrm{d}r.$$

Now let us assume that, as $\varepsilon \to 0$, $S(\boldsymbol{u})(x, \varepsilon)/\varepsilon$ tends to a limit $s(\boldsymbol{u})(x)$. Then

$$D_\varepsilon(\boldsymbol{u}) \to \tfrac{1}{4} s(\boldsymbol{u}) \int_0^\infty \varphi'(r)\, r^3 \, \mathrm{d}r = -\frac{3}{16\pi} s(\boldsymbol{u}).$$

The four-fifth law (von Karman and Howarth, Kolmogorov) says that for a stationary, homogeneous and isotropic random turbulent velocity field $\boldsymbol{u}$ one should have

$$\left\langle \left(\delta\boldsymbol{u} \cdot \frac{\xi}{|\xi|} \right)^3 \right\rangle = -\tfrac{4}{5} D|\xi|$$

where D is the mean rate of (inertial) energy dissipation per unit mass and $\langle\,\rangle$ denotes the statistical mean.

Without isotropy, one proves (Monin, cf Frisch [3])

$$D = -\tfrac{1}{4} \operatorname{div}_\xi \langle (\delta\boldsymbol{u})^2 \delta\boldsymbol{u} \rangle \big|_{\xi=0}$$

integrating in ξ over the ball $|\xi| \leqslant \varepsilon$ one obtains

$$D = -\frac{3}{16\pi} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\langle \int_{|\xi|=1} (\boldsymbol{u}(x + \varepsilon\xi) - \boldsymbol{u}(x))^2 (\boldsymbol{u}(x + \varepsilon\xi) - \boldsymbol{u}(x)) \cdot \xi \, \mathrm{d}\Sigma(\xi) \right\rangle.$$

Our expression of

$$s(u) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{|\xi|=1} (\boldsymbol{u}(x+\varepsilon\xi) - \boldsymbol{u}(x))^2 (\boldsymbol{u}(x+\varepsilon\xi) - \boldsymbol{u}(x)) \cdot \xi \, \mathrm{d}\Sigma(\xi)$$

thus simply gives a local non-random form of the above expression of the inertial dissipation.

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