Exercise Sheet 1

1. (Shear flows). Show that any shear flow $u(t, x_1, x_2, x_3) = f(x_1, x_2)e_3$ as defined in class is indeed a wewak solution to the incompressible Euler equations with zero pressure.

2. (Helicity). Let $u : (0, T) \times \mathbb{T}^3 \to \mathbb{R}^3$ be the velocity field of a smooth solution of the incompressible Euler equations. Show that the *helicity* defined as
$$H(t) := \int_{\mathbb{T}^3} u(t, x) \cdot \omega(t, x)\, dx$$
is conserved! Here $\omega = \nabla \times u$ is the associated vorticity.

3. For $0 \leq s < 1$ and $1 \leq p < \infty$, show that $B^s_{\infty,\infty}(\mathbb{T}^n) = C^\alpha(\mathbb{T}^n)$ and $B^s_{p,p}(\mathbb{T}^n) = W^{s,p}(\mathbb{T}^n)$.

4. Prove the Interpolation theorem for Besov spaces.

5. Prove the 'cool fact' presented in class. Is this fact true if we replace the set of positive Lebesgue measure $\mathcal{L}^n$ with a set $A \subset \mathbb{R}^n$ satisfying $\mathcal{H}^{n-\epsilon}(A) > 0$?

6. Show that for any set $A \subset \mathbb{R}^n$, $\dim_H(A) \leq \dim_{\underline{M}}(A) \leq \dim_{\overline{M}}(A)$. Give examples to show that the inequalities can be strict.

7. (Self-similar sets). A similarity transformation with parameter r is a map $T : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$
$$|T(x) - T(y)| = r|x - y|\,.$$
For $i = 1, ..., N$ let T_i be similarity transformations with parameter r_i where $0 < r_i < 1$. A self-similar set is a set $A \subset \mathbb{R}$ such that
$$A = \bigsqcup_{i=1}^{N} T_i(A)\,.$$
Let s be the unique number so that $\sum_{i=1}^N r_i^s = 1$. Show that
$$\dim_H(A) = \dim_{\overline{M}}(A) = s\,.$$