J. Math. Pures Appl. 79, 6 (2000) 525–552

ON THE FRISCH–PARISI CONJECTURE

Stéphane JAFFARD [1]

Département de Mathématiques, Université Paris XII, Faculté des Sciences et Technologie, 61 Av. du Gal. de Gaulle, 94010 Créteil Cedex, France

Received in 19 January 2000

ABSTRACT. – We prove several related results concerning the genericity (in the sense of Baire's categories) of multifractal functions. One result asserts that, if $s - d/p > 0$, quasi-all functions of the Sobolev space $L^{p,s}(\mathbf{R}^d)$ (or the Besov space $B_p^{s,q}(\mathbf{R}^d)$) are multifractal functions, with a spectrum of singularities supported by the interval $[s - d/p, s]$, on which the spectrum is $d(H) = d - (s - H)p$. Another result asserts that the Frisch–Parisi conjecture also holds for quasi-all functions, if the range of ps over which one computes the Legendre transform is chosen appropriately. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The Frisch–Parisi conjecture concerns the multifractal properties of functions that belong to some function spaces. Therefore, in order to state this conjecture, we first need to recall some basic definitions concerning the multifractal analysis of functions. We start with the definition of *pointwise* Hölder regularity $C^\alpha(x_0)$. Let $x_0 \in \mathbf{R}^d$ and let α be a positive real number. A function $f(x):\mathbf{R}^d \to \mathbf{R}$ is $C^\alpha(x_0)$ if there exists a constant $C > 0$ and a polynomial P_{x_0} of degree at most $[\alpha]$ such that in a neighbourhood of x_0,

$$\big|f(x) - P_{x_0}(x)\big| \leqslant C|x - x_0|^\alpha. \tag{1}$$

Note that this definition is local and involves no uniform regularity. The *Hölder exponent* of f at x_0 is

$$h_f(x_0) = \sup\big\{\alpha:\ f \in C^\alpha(x_0)\big\}.$$

Multifractal analysis is concerned in the study of the (usually fractal) sets S_H where a function f has a given Hölder exponent H. The domain of definition of the spectrum of singularities $d(H)$ is the set of values of H such that S_H is not empty. If H belongs to this domain of definition, $d(H)$ is the Hausdorff dimension of S_H ($d(H)$ is often called the Hölder spectrum). The function $d(H)$ can be extended to the whole real line by using the convention $\dim(\emptyset) = -\infty$, so that $d(H) = -\infty$ if H is nowhere the Hölder exponent of f (this convention is consistent with the Legendre transform approach that we will describe, since this approach is expected to yield $-\infty$ for the values of H that are not an Hölder exponent of f). A function

[1] E-mail: jaffard@univ-paris12.fr

is called multifractal when its spectrum of singularities is defined at least on an interval of non-empty interior.

Multifractal analysis started to be developed in the context of fully developed turbulence. B. Mandelbrot first introduced cascade models for the dissipation of energy in a turbulent fluid, see [22,23] and [20], that turned out to be multifractal measures, see [5] and references therein. This remarkable insight did meet the experimental results obtained in wind-tunnels which show that the regularity of the velocity of a turbulent fluid fluctuates widely from point to point, see [12]. This phenomenon, related to intermittency, suggests that the spectrum of singularities of the velocity of the fluid might be a universal function, in which case its determination would yield a fundamental information on the nature of turbulence. Obviously, it is almost impossible to obtain numerically a spectrum of singularities from its mathematical definition since it involves the successive determination of several intricate limits. Uriel Frisch and Giorgio Parisi proposed to derive the spectrum of singularities from 'averaged quantities' (the computation of which should be numerically stable) extracted from the signal; in [11] they proposed the following formula using the L^p modulus of continuity of the velocity; let

$$S_p(l) = \int \left| f(x+l) - f(x) \right|^p \mathrm{d}x. \tag{2}$$

Suppose now that $S_p(l)$ scales like $|l|^{\zeta_f(p)}$ when $l \to 0$ ($\zeta_f(p)$ is therefore called the *scaling function* of f); Frisch and Parisi conjectured that the spectrum of singularities can be obtained using the formula:

$$d(H) = \inf_p \bigl(pH - \zeta_f(p) + d\bigr), \tag{3}$$

see [11] or [14] for the heuristical derivation of (3) using similarities with statistical physics. Clearly, the domain of validity of this formula cannot be arbitrary since, for instance, $S_p(l)$ only involves first order differences, and therefore is not expected to give information on Hölder exponents larger than 1. This restriction and several similar ones can be withdrawn using the relation between $S_p(l)$ and Sobolev or Besov-type norms; indeed, if $p \geqslant 1$, and $\zeta_f(p) \in [0,1]$, $\zeta_f(p) = \sup\{\tau\colon\ f \in B^{\tau/p,\infty}_{p,\mathrm{loc}}\}$, see [14]. (We use $B^{s,\infty}_{p,\mathrm{loc}}$ instead of $B^{s,\infty}_p$ because the functions considered are not expected to have any decay at infinity, and the integral (2) is computed on bounded domains.) Since Besov spaces are defined for arbirary values of s and p, the function

$$\eta_f(p) = \sup\bigl\{\tau\colon\ f \in B^{\tau/p,\infty}_{p,\mathrm{loc}}\bigr\} \tag{4}$$

is thus a natural extension of $\zeta_f(p)$ which is defined without any restriction on its domain of definition (as long as p is positive) or on its range; we will therefore also call $\eta_f(p)$ the scaling function of f. It is thus natural to conjecture that:

$$d(H) = \inf_p \bigl(pH - \eta_f(p) + d\bigr). \tag{5}$$

When (5) holds we say that the *multifractal formalism* is satisfied. Of course we should state precisely the range of ps on which the infimum is taken; it is usually assumed that the infimum has to be taken on all positive ps. We will see that it is not the case: The validity of formula (5) depends precisely on a right choice for this range which must be smaller as we will see in the following.

Remarks. – Since formula (5) is a Legendre transform, it can hold only for spectra that are concave, and since the function $\eta_f(p)$ is defined only for positive ps, it can yield only the increasing part of a $\bigcap$-shaped spectrum.

The Frisch–Parisi formula has been extended to negative ps, and in that case, it is expected to yield the right-side, decreasing part of a $\bigcap$-shaped spectrum; one has to renormalize the possible divergence of the integral (2), which can be done using wavelet techniques (see [3] for the numerical technique of the Wavelet Maxima Method and [17] for a mathematical framework which yields a natural extension of Besov spaces to negatives ps). However, these extensions do not correspond to inclusions in topological vector spaces and therefore the problem solved in the present paper cannot be formulated in this setting.

Note that the initial examples of Mandelbrot were multifractal measures; in this context, the Hölder exponent at x_0 has to be replaced by the local dimension at x_0, defined as:

$$\liminf_{r\to 0} \frac{\log \mu(B(x_0, r))}{\log r},$$

(where $B(x_0, r)$ denotes the ball of center x_0 and radius r). The quasi-sure results of [6] concern one-dimensional measures.

Though the conjecture of Frisch and Parisi was stated in the context of fully developed turbulence, the heuristic argument used in its derivation does not use any specific assumption on turbulent flows. The validity of (5) can therefore be raised in a more general context; it is far from holding for all functions in a given function space; indeed it is extremely easy to construct counterexamples to (5). On the opposite, each time (5) has been shown to hold, it was the consequence of a functional equation satisfied by the function under study (usually a selfaffinity property, either exact, approximate, or stochastic). Therefore the general consensus among mathematicians and physicists was that the validity of the multifractal formalism must be the consequence of a precise inner structure of the function considered. The purpose of the present paper is to show that the opposite is true: The Frisch–Parisi conjecture holds for quasi-all functions, i.e. outside a set of the first class of Baire. Let us explain more precisely what we mean.

The Frisch–Parisi conjecture, reformulated as in (5) states that, if f belongs to the topological vector space

$$V = \bigcap_{\varepsilon>0,\ p>0} B_{p,\mathrm{loc}}^{(\eta(p)-\varepsilon)/p,\, p} \tag{6}$$

then its spectrum of singularities satisfies (5).

We will see that V is a Baire's space, i.e. that any countable intersection of everywhere dense open sets is everywhere dense; we will show that in the space V, the set of functions that satisfy (5) contains a countable intersection of everywhere dense open sets of V, i.e. contains a dense G_δ set; we will use the traditional expressions: (5) holds generically in V, or quasi-all functions of V satisfy (5). In order to state precisely our result, we first have to determine which functions $\eta(p)$ can be written as (4), and to specify the range of ps in (5).

The properties of a scaling function $\eta(p)$ are more easily expressed using an auxiliary function $s(q)$ defined with the help of the *Besov domain*. The Besov domain B_f of a function f is simply the set of (q, s) such that f belongs to $B_{1/q,\mathrm{loc}}^{s,1/q}$. By interpolation, the Besov domain has to be a convex subset of $\mathbf{R}^2$, and the Besov embeddings imply that, if (q, s) belongs to B_f, then the segment:

$$\{(t, d(t-q)+s),\ t \in]0, q[\}$$

also belongs to B_f, see [27]. It follows that the boundary of the Besov domain is the graph of a function $s(q)$ which is concave and, since $s(q)$ is clearly increasing,

$$0 \leqslant s'(q) \leqslant d. \tag{7}$$

Note that, in the definition of the Besov domain, we can use indifferently any of the spaces $B^{s,r}_{1/q,\mathrm{loc}}$ (for an arbitrary r) instead of $B^{s,1/q}_{1/q,\mathrm{loc}}$; such a choice doesn't change the boundary of the Besov domain, hence, it wouldn't change $s(q)$. When $q \leqslant 1$, we can also use the Sobolev space $L^{1/q,s}_{\mathrm{loc}}$.

PROPOSITION 1. – *Any concave function $s(q)$ satisfying* (7) *defines the Besov domain of a distribution f.*

Such distributions with a prescribed Besov domain will be constructed in Section 3.1. From the definition of $\eta(p)$ it follows that $s(q) = q\eta(1/q)$, and conversely $\eta(p) = ps(1/p)$. We will need furthemore that the distributions f we consider are actually functions with some uniform regularity, i.e. that there exists $\gamma > 0$ such that $f \in C^\gamma(\mathbf{R}^d)$. This condition can be written $s(0) > 0$. Let us introduce the following definitions:

DEFINITION 1. – *A function $\eta(p): \mathbf{R}^+ \to \mathbf{R}$ is said to be admissible if $s(q) = q\eta(1/q)$ is concave and satisfies $0 \leqslant s'(q) \leqslant d$. It is strongly admissible if furthermore $s(0) > 0$.*

One immediately checks that if $\eta(p)$ is admissible, it is concave.

It is proved in [14] that, if $s > d/p$, the spectrum of singularities of every function of $B^{s,q}_p$ satisfies:

$$d(H) \leqslant d - p(s - H) \quad \text{for } H \geqslant s - d/p \tag{8}$$

and

$$d(H) = -\infty \quad \text{for } H < s - d/p \tag{9}$$

(note that the similar result concerning measures was previously proved in [7]). If the scaling function $\eta_f(p)$ of a function f is strongly admissible, it follows from the concavity of $s(q)$ that there exists a critical value of p, denoted by p_c, such that if $q < 1/p_c$, $s(q) > dq$ and if $q > 1/p_c$, $s(q) < dq$ (except in the degenerate case where $s(q) = s_0 + dq$ in which case $p_c = +\infty$). We can apply (8) for every (s, p) such that $s > d/p$ and $f \in B^{s,q}_p$; this can be done only for $p > p_c$. It follows that:

$$d(H) \leqslant \inf_{p \geqslant p_c} \bigl(pH - \eta(p) + d\bigr). \tag{10}$$

Since no upper bound holds in general for $p \leqslant p_c$ (see [14]), this formula suggests that, in the Frisch–Parisi conjecture, the right range of ps on which the Legendre transform has to be calculated is $p \in [p_c, +\infty)$. The following theorem, proved in Section 3, shows that it is indeed the case.

THEOREM 1. – *Let $\eta(p)$ be a strongly admissible function and V be the function space defined by* (6). *The domain of definition of the spectrum of singularities of quasi-all functions of V is the interval $[s(0), d/p_c]$ where it is given by*:

$$d(H) = \inf_{p \geqslant p_c} \bigl(Hp - \eta(p) + d\bigr). \tag{11}$$

Furthermore, for quasi-all functions of V, the Hölder exponent takes almost everywhere the value d/p_c.

If $\eta(p)$ is only admissible, but not strongly admissible, quasi-all functions of V are not locally bounded, so that their spectra of singularities are nowhere defined whatever H be.

Remarks. – Formula (11) states that the spectrum of quasi-all functions is composed of two parts:

- a part defined by $H < \eta'(p_c)$ where the infimum in (11) is attained for $p > p_c$, and the spectrum can be computed as the 'usual' Legendre transform of $\eta(p)$

$$d(H) = \inf_{p>0}\big(Hp - \eta(p) + d\big);$$

- a part defined by $\eta'(p_c) \leqslant H \leqslant d/p_c$ where the infimum in (11) is attained for $p = p_c$, and the spectrum is a straight line

$$d(H) = Hp_c.$$

This second case is rather unexpected, and shows that the Frisch–Parisi conjecture fails in this part of the spectrum; we will explain in the following the reason of thus failure.

Comparing (10) and (11) we see that quasi-all function of V strive to have their Hölder singularities on a set as large as possible.

Consequences concerning the quasi-sure validity of the multifractal formalism for measures cannot be directly deduced from Theorem 1.

The study of the properties of quasi-all functions with a given a priori regularity goes back to the famous papers of Banach [4], Mazurkiewicz [24], Jarnik [19] and Saks [28] at the beginning of the 30's, which give differentiability properties of quasi-all continuous functions. At the beginning of the 80's, differentiability properties of monotone continuous functions were studied by T. Zamfirescu in [29] and [30]; this line of research recently culminated in the work of Z. Buczolich and J. Nagy who proved in [6] that quasi-all monotone continuous functions are multifractal with spectrum $d(H) = H$ for $H \in [0, 1]$. Their paper was the starting point of the present one.

In Section 2 we will study a related but simpler problem; namely, we will prove that quasi-all functions in a given Besov or Sobolev space are multifractal with a given spectrum, and we will determine this generic spectrum. Note that Sobolev spaces are Baire spaces, and Besov spaces are Baire spaces since they are metric spaces (or pseudometric spaces if $p < 1$) and complete, see [27].

THEOREM 2. – *Let $p > 0$, $q > 0$ and $s > d/p$. The domain of definition of the spectrum of singularities of quasi-all functions of $B_p^{s,q}(\mathbf{R}^d)$ is the interval $[s - d/p, s]$ where it is given by*

$$d(H) = p(H - s) + d. \tag{12}$$

Furthermore, for quasi-all functions of $B_p^{s,q}(\mathbf{R}^d)$, the Hölder exponent takes almost everywhere the value s. If $p > 1$, the same result holds for $L^{p,s}(\mathbf{R}^d)$.

If $s < d/p$, quasi-all functions of $B_p^{s,q}(\mathbf{R}^d)$ or of $L^{p,s}(\mathbf{R}^d)$ are not locally bounded, so that their spectra of singularities are defined for no value of H.

If $s < d/p$ and $\gamma > 0$, the domain of definition of the spectrum of singularities of quasi-all functions of $B_p^{s,q}(\mathbf{R}^d) \cap C^\gamma(\mathbf{R}^d)$ or of $L^{p,s}(\mathbf{R}^d) \cap C^\gamma(\mathbf{R}^d)$ is the interval $[\gamma, \frac{d\gamma}{d-sp+\gamma p}]$ where their spectrum is given by:

$$d(H) = \left(p + \frac{d - sp}{\gamma}\right)H. \tag{13}$$

Furthermore, for quasi-all functions of $B_p^{s,q}(\mathbf{R}^d)\cap C^\gamma(\mathbf{R}^d)$ or of $L^{p,s}(\mathbf{R}^d)\cap C^\gamma(\mathbf{R}^d)$, the Hölder exponent takes almost everywhere the value $\frac{d\gamma}{d-sp+\gamma p}$.

This theorem shows that quasi-all functions in a given Besov or Sobolev space are multifractal, except when $p=\infty$ where the spectrum is reduced to one point (in that case the function is called monofractal). Apart from its own interest, Theorem 2 can be seen as a first step towards the proof of Theorem 1 for two reasons; first, we will see, as a consequence of Proposition 4 that the multifractal formalism holds for quasi-all functions in a given Besov or Sobolev space, so that Theorem 2 can be seen as a particular case of Theorem 1. The second reason is that the proof of Theorem 2 will allow us to introduce the tools needed afterwards in the general framework of the Frisch–Parisi conjecture in a much simpler setting, since we have to deal with only one space at a time. Furthermore the inspection of the last case confirms what is the right range of ps needed in the statement of the Frisch–Parisi conjecture.

It follows from (8) and (9) that, in the first assertion of Theorem 2, we only have to prove that $d(H)$ is larger than $p(H-s)+d$, and that, if $H>s$, H is not an Hölder exponent.

Similarly, it was also proved in [18] that, if $s<d/p$, the spectrum of singularities of every function of $B_p^{s,q}\cap C^\gamma$ satisfies:

$$d(H)\leqslant\left(p+\frac{d-sp}{\gamma}\right)H\quad\text{for } H\in\left[\gamma,\frac{d\gamma}{d-sp+\gamma p}\right] \tag{14}$$

and

$$d(H)=-\infty\quad\text{for } H<\gamma. \tag{15}$$

It follows that, in the last assertion of Theorem 2, we only have to prove that $d(H)$ is larger than $(p+\frac{d-sp}{\gamma})H$ and that, if $H>\frac{d\gamma}{d-sp+\gamma p}$, H is not an Hölder exponent.

2. Multifractal analysis of quasi-all functions of $B_p^{s,q}(\mathbf{R}^d)$ and $L^{p,s}(\mathbf{R}^d)$

Firts note that the spaces we consider in this section are either Banach or quasi-Banach spaces (see [27]) so that they are Baire spaces. Our main tool for proving Theorem 1 and Theorem 2 is orthonormal wavelet decompositions. Let us mention at this point that the idea of using wavelet techniques in multifractal analysis has been worked out first by Alain Arneodo and his coworkers, see for instance [3] and references therein. We start by recalling some notations and properties of wavelet expansions.

Let $(\psi^{(i)})_{i=1,\dots,2^d-1}$ be wavelets in the Schwartz class as constructed in [21]. The functions

$$2^{dj/2}\psi^{(i)}(2^jx-k),\quad j\in\mathbf{Z},\ k\in\mathbf{Z}^d,$$

form an orthonormal basis of $L^2(\mathbf{R}^d)$. We index the wavelets in terms of the dyadic cubes: If λ is the cube

$$\lambda=\left\{x\in\mathbf{R}^d\colon 2^jx-k\in[0,1]^d\right\},$$

we use the notation $\psi_\lambda^{(i)}(x)=\psi^{(i)}(2^jx-k)$; thus

$$f(x)=\sum_{i,\lambda}c_\lambda^{(i)}\psi_\lambda^{(i)}(x), \tag{16}$$

where the wavelet coefficients of f are given by

$$c_\lambda^{(i)} = \int_{\mathbf{R}^d} 2^{dj} \psi^{(i)}\big(2^j t - k\big) f(t)\, \mathrm{d}t.$$

(Note that we do not use the usual L^2 normalization; the natural normalization for the problem we consider is the L^∞ normalization.)

Wavelets supply an efficient tool to study the Frisch–Parisi conjecture for two reasons: First, they yield characterizations of Besov and Sobolev spaces, [25]: In view of the L^∞ normalization

$$f \in B_p^{s,q} \iff \left(\sum_k \big| c_\lambda^{(i)} 2^{(s-\frac{d}{p})j} \big|^p \right)^{1/p} = \varepsilon_j \quad \text{with } \varepsilon_j \in l^q, \tag{17}$$

$$f \in L^{p,s} \iff \left(\sum_{\lambda,i} \big| c_\lambda^{(i)} \big|^2 2^{2sj} \chi_\lambda(x) \right)^{1/2} \in L^p, \tag{18}$$

where $\chi_\lambda(x)$ denotes the characteristic function of the set λ. In particular, $f \in C^\gamma(\mathbf{R}^d)$ if the sequence $c_\lambda^{(i)} 2^{\gamma j}$ belongs to l^∞.

Similarly, if $f \in C^\gamma(\mathbf{R}^d)$ for a $\gamma > 0$, it is shown in [13] that the Hölder exponent of f can be computed at every point by the formula:

$$h_f(x_0) = \liminf_{\lambda \to x_0} \frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)}. \tag{19}$$

Thus, the quantities that appear in formula (5) can all be expressed in terms of wavelet coefficients.

The idea of the proof of Theorem 2 begins with the construction of functions for which the equalities (12) or (13) hold. We will call such functions *saturating functions* because they 'saturate' the inequalities (10). Then, we start with a sequence f_n dense in the Besov or Sobolev space we consider. We slightly perturbate f_n by replacing its wavelet coefficients for $j \geqslant j_n$ by those of the saturating function. We obtain a new dense sequence g_n which satisfy (5). The result is obtained by considering a residual set of the form:

$$A = \bigcap_{N \in \mathbf{N}} \bigcup_{n \geqslant N} B(g_n, r_n),$$

where $B(g_n, r_n)$ denotes the open ball (using the norms (17) or (18)) of center g_n and radius r_n; the r_n are chosen small enough so that for $j = j_n$, the wavelet coefficients of the functions of $B(g_n, r_n)$ are 'close' to those of g_n so that the spectra of the functions of A will be larger than $p(H - s) + d$.

2.1. Saturating functions when $s > d/p$

We consider a given Besov space $B_p^{s,q}(\mathbf{R}^d)$ where $s > d/p$, $p \neq +\infty$ and $q \neq +\infty$. In this subsection we construct and study the multifractal properties of one specific saturating function F adapted to this space. We will obtain that the spectrum of F satisfies (12) when the Hölder exponent is computed for $x_0 \in (0,1)^d$; it will follow that the spectrum of quasi-all functions is also given by (12) when the Hölder exponent is computed for $x_0 \in (0,1)^d$; clearly this result does not depend on the particular choice of the unit cube, so that it is true for any open cube. Covering $\mathbf{R}^d$ by a countable family of open cubes, we will obtain (12) for a countable intersection of dense G_δ sets, hence on a G_δ set. Thus, from now on, we work on $(0,1)^d$. We now define the wavelet coefficients $c_\lambda^{(i)}$ of F. Let $j \geqslant 1$ and $k \in \{0, \dots, 2^j - 1\}$ be given. We define $J \leqslant j$ as follows: Consider the irreducible representation

$$\frac{k}{2^j} = \frac{K}{2^J}, \quad \text{where } K \in \mathbf{Z}^d - (2\mathbf{Z})^d. \tag{20}$$

Let $a \geqslant 1$ be a real exponent; we choose

$$c_\lambda^{(i)} = \frac{1}{j^a} 2^{(\frac{d}{p} - s)j} 2^{-\frac{d}{p}J}. \tag{21}$$

Note that the term $2^{-\frac{d}{p}J}$ is responsible for the strong variations of regularity of F; if we take it off, the function thus constructed is a Weierstrass function, which has a constant Hölder exponent.

PROPOSITION 2. – *If*

$$a = \frac{2}{p} + \frac{2}{q} + 1, \tag{22}$$

the saturating function F whose wavelet coefficients are given by (21) *belongs to $B_p^{s,q}$.*

The domain of definition of its spectrum of singularities is the interval $[s - \frac{d}{p}, s]$, where

$$d(H) = p(H - s) + d.$$

The Hölder exponent of F takes the value s almost everywhere.

We start by proving the first part of Proposition 2. Let j be given. For each $J \leqslant j$ there are less than 2^{dJ} values of k satisfying (20); thus

$$\sum_k \left|c_\lambda^{(i)} 2^{(s - \frac{d}{p})j}\right|^p \leqslant \frac{1}{j^{ap}} \sum_{J=0}^{j} 2^{dJ} \left(2^{-\frac{d}{p}J}\right)^p = j^{1-ap}$$

and (17) will be satisfied if

$$\sum_{j \geqslant 1} \left(j^{\frac{1}{p} - a}\right)^q < \infty, \tag{23}$$

so that we can choose $a = \frac{2}{p} + \frac{2}{q} + 1$. Let us now determine the Hölder exponent of this saturating function everywhere on $(0,1)^d$; it will depend on the dyadic approximation properties of the point considered.

DEFINITION 2. – *A point $x_0 \in \mathbf{R}^d$ is α-approximable by dyadics if there exists a sequence (k_n, j_n) such that:*

$$\left| x_0 - \frac{k_n}{2^{j_n}} \right| \leqslant \frac{1}{2^{\alpha j_n}}. \tag{24}$$

The dyadic exponent of x_0 is the supremum of all αs such that x_0 is α-approximable by dyadics. We denote it by $\alpha(x_0)$.

The dyadic exponent of a point is of course never smaller than 1.

LEMMA 1. – *The Hölder exponent of the saturating function F is*

$$h_F(x_0) = s - \frac{d}{p} + \frac{d}{\alpha(x_0)p}.$$

Proof of Lemma 1. – Let x_0 be fixed. For each wavelet coefficient we will estimate the order of magnitude of

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)}; \tag{25}$$

indeed, since $F \in C^\gamma$ for a $\gamma > 0$, using (19), it will yield the Hölder exponent of F at x_0.

First, we obtain an upper bound for the Hölder exponent. Let $\varepsilon > 0$. There exists an infinite number of (K_n, J_n) such that:

$$\left| x_0 - \frac{K_n}{2^{J_n}} \right| \leqslant \frac{1}{2^{(\alpha(x_0)-\varepsilon)J_n}}. \tag{26}$$

Consider the wavelet coefficients $c_\lambda^{(i)}$ such that $k/2^j = K_n/2^{J_n}$ and $j = [\alpha(x_0)J_n]$. Since

$$c_\lambda^{(i)} = \frac{1}{j^a} 2^{(\frac{d}{p}-s)j} 2^{-\frac{d}{p}J_n},$$

it follows that

$$\begin{aligned}\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)} &= \frac{\log(|c_\lambda^{(i)}|)}{-j\log 2}\bigl(1 + \mathrm{o}(1)\bigr) \\ &= \left(s - \frac{d}{p} + \frac{J_n}{j}\frac{d}{p}\right)\bigl(1 + \mathrm{o}(1)\bigr) \\ &= \left(s - \frac{d}{p} + \frac{d}{\alpha(x_0)p}\right)\bigl(1 + \mathrm{o}(1)\bigr).\end{aligned}$$

Thus the liminf of (25) on these coefficients is $s - \frac{d}{p} + \frac{d}{\alpha(x_0)p}$ and the upper bound holds.

Now, we obtain a lower bound for the Hölder exponent. Let $\varepsilon > 0$, and j and k be given. We define J and K by $K/2^J = k/2^j$, where $K \in \mathbf{Z}^d - (2\mathbf{Z})^d$. Since $\alpha(x_0)$ is the dyadic exponent at x_0, for J large enough, we have:

$$\frac{1}{2^{(\alpha(x_0)+\varepsilon)J}} < \left| x_0 - \frac{K}{2^J} \right| = \left| x_0 - \frac{k}{2^j} \right|. \tag{27}$$

We separate two types of coefficients:

(1) *The coefficients $c_\lambda^{(i)}$ such that $|k/2^j - x_0| \leqslant 2^{-j}$.*

From the irreducibility of the fraction $K/2^J$, it follows that $j \geqslant J$, and from (27) it follows that

$$j \leqslant \bigl(\alpha(x_0) + \varepsilon\bigr)J. \tag{28}$$

Furthermore,

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)} = \frac{\log(|c_\lambda^{(i)}|)}{-j\log 2}\bigl(1 + \mathrm{o}(1)\bigr) = \left(s - \frac{d}{p} + \frac{J}{j}\frac{d}{p}\right)\bigl(1 + \mathrm{o}(1)\bigr). \tag{29}$$

Because of (28) and (29), the liminf of (25) taken on the coefficients of this first case is larger than

$$s-\frac{d}{p}+\frac{d}{p(\alpha(x_0)+\varepsilon)}.$$

(2) *The coefficients* $c_\lambda^{(i)}$ *such that* $|k/2^j - x_0| > 2^{-j}$.

It follows that

$$j\log 2 > -\log\bigl(|x_0-\lambda|\bigr); \tag{30}$$

thus

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j}+|x_0-\lambda|)}=\frac{\log(|c_\lambda^{(i)}|)}{\log(|x_0-\lambda|)}\bigl(1+\mathrm{o}(1)\bigr)=\frac{\bigl((s-\frac{d}{p})j+\frac{d}{p}J\bigr)\log 2}{-\log(|x_0-\lambda|)}\bigl(1+\mathrm{o}(1)\bigr).$$

Using (30), this is larger than

$$\left(s-\frac{d}{p}+\frac{d}{p}\,\frac{J\log 2}{-\log(|x_0-\lambda|)}\right)\bigl(1+\mathrm{o}(1)\bigr)$$

which, because of (27) exceeds

$$\left(s-\frac{d}{p}+\frac{d}{p(\alpha(x_0)+\varepsilon)}\right)\bigl(1+\mathrm{o}(1)\bigr). \tag{31}$$

Since ε can be chosen arbitrarily small, Lemma 1 follows. □

The computation of the spectrum of singularities of the saturating function F is now immediate using a standard result of dyadic approximation, see [10] for instance, which states that the Hausdorff dimension of the set of points with dyadic exponent α is exactly d/α. Thus, $d(H)=\frac{d}{\alpha}$ if $H=s-\frac{d}{p}+\frac{1}{\alpha}\frac{d}{p}$. Since α can take any value larger than 1, the second point of Proposition 2 follows. Furthermore, since almost every point has its dyadic exponent equal to 1, the last point of Proposition 2 holds.

2.2. The residual set for $B_p^{s,q}$ if $s-d/p>0$

We suppose first that $p\neq\infty$ and $q\neq\infty$. In that case, the space $B_p^{s,q}$ is separable and we can pick a sequence f_n dense in $B_p^{s,q}$. We denote by g_n the following modification of f_n:

- If $j<n$ the wavelet coefficients of g_n are the same as those of f_n.
- If $j\geqslant n$ the wavelet coefficients $c_\lambda^{(i)}$ of g_n are the same as those of the saturating function F constructed in the previous subsection.

It clearly follows from our choice of the wavelet coefficients of the saturating function F that $\|f_n-g_n\|$ in $B_p^{s,q}$ tends to 0 when n tends to $+\infty$. Thus the sequence g_n is also dense in $B_p^{s,q}$. The G_δ dense set that we will consider is:

$$A=\bigcap_{m\in\mathbf{N}}\bigcup_{n\geqslant m}B(g_n,r_n), \tag{32}$$

where $r_n=\frac{1}{2n^a}2^{-nd/p}$.

A is clearly a countable intersection of dense open sets. We have chosen r_n small enough so that, at the scale $j=n$ each wavelet coefficient of a function f in $B(g_n,r_n)$ is close to the corresponding coefficient of F; indeed, it follows from (21) that the smallest coefficient of the saturation function F at the scale $j = n$ is $\frac{1}{n^a} 2^{-sn}$ and, because of our choice of r_n, the corresponding wavelet coefficient of f is between half and two times this quantity.

If a function f belongs to the residual set A, it belongs to an infinite number of balls $B(g_n, r_n)$. Denote by B_{n_l} this sequence of balls. Thus, at the scales $j = n_l$, the wavelet coefficients of f are 'close' to those of the saturating function F. We denote by F_α the set of points x such that:

$$\exists K \in \mathbf{Z}^d - (2\mathbf{Z})^d \colon \left| x - \frac{K}{2^{[n_l/\alpha]}} \right| \leqslant \frac{1}{2^{n_l}}$$

for an infinite number of values of l. This set can also be defined as:

$$\limsup_{m \to \infty} \bigcup_{l \geqslant m, K} \frac{K}{2^{[n_l/\alpha]}} + \left[-\frac{1}{2^{n_l}}, \frac{1}{2^{n_l}} \right]^d.$$

Let us first estimate the Hölder exponent of f at such a point $x \in F_\alpha$. We consider the wavelet coefficent indexed by j and k such that $j = n_l$ and

$$\frac{k}{2^j} = \frac{K}{2^{[n_l/\alpha]}};$$

thus

$$c_\lambda^{(i)} \geqslant \frac{1}{j^a} 2^{(\frac{d}{p} - s) j} 2^{-\frac{d}{p}[n_l/\alpha]}$$

and

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)} = \frac{\left(\frac{d}{p} - s\right) j - \frac{d}{p}[n_l/\alpha]}{-j} \big(1 + \mathrm{o}(1)\big) = \left(s - \frac{d}{p} + \frac{d}{\alpha p}\right) \big(1 + \mathrm{o}(1)\big)$$

so that the Hölder exponent at this point is smaller than $s - \frac{d}{p} + \frac{d}{\alpha p}$. In order to compute the dimension of F_α, let us first recall the following definitions:

Let $h : \mathbf{R}^+ \to \mathbf{R}^+$ be a continuous increasing function satisfying $h(0) = 0$, and let A be a bounded subset of $\mathbf{R}^d$. If $|B|$ denotes the diameter of the set B, let

$$\mathcal{H}_\varepsilon^h(A) = \inf_{\mathcal{U}} \left\{ \sum_{(u_i) \in \mathcal{U}} h\big(|u_i|\big) \right\},$$

where the infimum is taken on all coverings $\mathcal{U}$ by families of balls $(u_i)_{i \in \mathbf{N}}$ of radius at most ε. The $\mathcal{H}^h$-measure of A is defined as:

$$\mathcal{H}^h(A) = \lim_{\varepsilon \to 0} \mathcal{H}_\varepsilon^h(A).$$

We use the functions $h_a(x) = (\log x)^2 |x|^a$.

Since

$$\bigcup_{m \geqslant l, K} \frac{K}{2^{[n_l/\alpha]}} + \left[-\frac{1}{2^{n_l/\alpha}}, \frac{1}{2^{n_l/\alpha}} \right]^d$$

is $[0,1]^d$, we are exactly in the position to apply Theorem 2 of [16] which asserts that $\mathcal{H}^{h_{d/\alpha}}(F_\alpha) > 0$ (so that the Hausdorff dimension of F_α is larger than d/α). Following Theorem 2.1 of [18], the set of points where a function in $B_p^{s,q}$ has a Hölder exponent less than $s - \frac{d}{p} + \frac{d}{\alpha p}$ has a vanishing $\mathcal{H}^{h_{d/\alpha}}$ Hausdorff measure. It follows that the set of points where the Hölder exponent is exactly $s - \frac{d}{p} + \frac{d}{\alpha p}$ has the dimension d/α.

Note that $F_1 = [0,1]^d$ and, if $x \in F_1$, $H(x) \leqslant s$; thus Theorem 2.1 of [18] implies that Hölder exponent is almost everywhere s.

We consider now the case where $p = q = \infty$, i.e. the $C^s(\mathbf{R}^d)$ case. Since $C^s(\mathbf{R}^d)$ is not separable, the argument in this case is slightly different. We still use a given wavelet basis, and we denote by E_m the set of functions whose coefficients $c_\lambda^{(i)}$ are each a nonvanishing multiple of $2^{-sj}/2^m$ on this basis. We choose for norm on C^s

$$\|f\| = \sup_{i,\lambda} \left| c_\lambda^{(i)} 2^{sj} \right|$$

and we define

$$A_m = E_m + B\left(0, \frac{1}{2^{m+1}}\right)$$

and

$$A = \bigcap_M \left(\bigcup_{m \geqslant M} A_m \right).$$

The set A is a countable intersection of open dense sets and, if f belongs to one A_m, its wavelet coefficients $d_\lambda^{(i)}$ satisfy:

$$\left| d_\lambda^{(i)} \right| \geqslant \frac{2^{-sj}}{2^{m+1}}$$

so that its Hölder exponent is everywhere equal to s.

We leave the case where only one among p and q is infinite as an exercise.

2.3. The Sobolev case

The $L^{p,s}$ case is obtained by a slight adaptation of the Besov case, so that we only mention the modifications.

Since $B_p^{s,1} \hookrightarrow L^{p,s} \hookrightarrow B_p^{s,\infty}$, we can choose for saturating function the one we constructed for $B_p^{s,1}$.

We pick for radius of the ball $B(g_n, r_n)$, $r_n = \frac{1}{2n^3} 2^{-nd/p}$, which, as in the Besov case insures that at the scale $j = n$ all wavelet coefficents of the elements of $B(g_n, r_n)$ are a slight modification of those of the saturating function F. The result follows as above.

2.4. The case $s - d/p < 0$

When $s - d/p < 0$, the upper bound (8) for the spectrum of singularities no more holds because some functions of $B_p^{s,q}$ or of $L^{p,s}$ are not locally bounded. We will now check that this is the case for quasi-all functions.

We suppose now that the wavelets we use are compactly supported, as in [8]. We will use the following lemma which is an immediate consequence of the formula defining the wavelet coefficients.

LEMMA 2. – *If f is bounded in a neighbourhood of x_0, there exist $r, C > 0$ and $J \in \mathbf{N}$ such that if $|\lambda - x_0| \leqslant r$ and $j \geqslant J$, $|c_\lambda^{(i)}| \leqslant C$.*

We pick for saturating function the function F whose wavelet coefficients are defined as follows: the coefficients $c_\lambda^{(i)}$ of F vanish except when each coordinate k_i of $k = (k_1, \dots, k_d)$ is a multiple of $[2^j/j]$, and in this case, $c_\lambda^{(i)} = j$.

There are less than j^d nonvanishing wavelet coefficients in the unit cube $[0,1]^d$, so that, if $\omega = d/p - s$,

$$\left(\sum_k \left|c_\lambda^{(i)} 2^{(s-\frac{d}{p})j}\right|^p\right)^{1/p} \leqslant j^{1+d/p} 2^{-\omega j} \tag{33}$$

and (17) holds.

The construction of the residual set follows the previous similar proofs. We pick a dense sequence f_n in $B_p^{s,q}$, we define g_n as in the beginning of Section 2.2, and the open ball around g_n is chosen of radius $r_n = \frac{n}{2} 2^{(s-\frac{d}{p})n}$. Thus, if f belongs to this ball, the wavelet coefficients of f indexed by $j = n$ and k such that each coordinate k_i of k is a multiple of $[2^j/j^2]$ are larger than $j/2$. We define as before:

$$A = \bigcap_{m \in \mathbf{N}} \bigcup_{n \geqslant m} B(g_n, r_n).$$

A function f that belongs to A belongs to an infinite number of balls $B(g_n, r_n)$, and thus cannot be locally bounded because of Lemma 2. Hence the second part of Theorem 2 holds (the proof for $L^{p,s}$ is exactly similar).

2.5. The case $B_p^{s,q} \cap C^\gamma$

We prove now the last part of Theorem 2; thus we consider the case of the intersection $B_p^{s,q} \cap C^\gamma(\mathbf{R}^d)$, and of course we suppose that $s > \gamma$ (because, if $s \leqslant \gamma$, the functions of C^γ belong locally to $B_p^{s,q}$).

We start by defining a saturating function F adapted to this case. Let

$$L = \left[\frac{d + (\gamma - s)p}{d} j\right]. \tag{34}$$

The wavelet coefficients of F are picked as follows:

$$c_\lambda^{(i)} = \begin{cases} j^{-2/q} 2^{-\gamma j} & \text{if each coordinate of } k \text{ is a multiple of } 2^j 2^{-L}, \quad (35) \\ j^{-2/q} 2^{-sj} & \text{else.} \quad (36) \end{cases}$$

PROPOSITION 3. – *The saturating function F defined by* (35) *and* (36) *belongs to $B_p^{s,q} \cap C^\gamma$. The domain of definition of its spectrum of singularities is the interval $[\gamma, \frac{\gamma d}{d+(\gamma-s)p}]$, where*

$$d(H) = \frac{d + (\gamma - s)p}{\gamma} H,$$

and for almost every x,

$$h_F(x) = \frac{d\gamma}{d - sp + \gamma p}.$$

Proof of Proposition 3. – The saturating function F belongs to $B_p^{s,q}$ because there are 2^{dL} wavelet coefficients of size $2^{-\gamma j}$ in the cube $[0,1]^d$ so that:

$$\left(\sum_k \left|c_\lambda^{(i)}\right|^p 2^{(sp-d)j}\right)^{1/p} \leqslant j^{-2/q}\left(2^{dL}\left(2^{-\gamma j}\right)^p 2^{(sp-d)j} + 2^{dj}\left(2^{-sj}\right)^p 2^{(sp-d)j}\right)^{1/p} \leqslant 2^{1/p} j^{-2/q};$$

F belongs to C^γ because $|c_\lambda^{(i)}| \leqslant 2^{-\gamma j}$.

Now, we calculate the Hölder exponent of F. By analogy with Section 2.1, we say that x is α-approximable if there exists an infinite number of wavelet coefficients $c_\lambda^{(i)}$ satisfying (35) and such that

$$2^{-j} + |x-\lambda| \leqslant 2^{-\alpha L}. \tag{37}$$

We define the exponent of approximation at x as the supremum of all such αs, and J_α as the set of points where the exponent of approximation is α. It follows that the exponent of approximation belongs to the interval:

$$\left[1, \frac{d}{d+(\gamma-s)p}\right].$$

Let $x \in J_\alpha$; $\forall \varepsilon > 0$, there exists an infinite number of wavelet coefficients $c_\lambda^{(i)}$ satisfying (35) such that $2^{-j} + |x-\lambda| \leqslant 2^{-(\alpha-\varepsilon)L}$. For these coefficients, (25) is smaller than

$$\frac{\gamma j}{(\alpha-\varepsilon)L} = \frac{\gamma d}{(\alpha-\varepsilon)(d+(\gamma-s)p)}. \tag{38}$$

For all wavelet coefficients satisfying (35) and such that j is large enough,

$$2^{-j} + |x-\lambda| \leqslant 2^{-(\alpha+\varepsilon)L},$$

so that (25) is larger than

$$\frac{\gamma j}{(\alpha+\varepsilon)L} = \frac{\gamma d}{(\alpha+\varepsilon)(d+(\gamma-s)p)}.$$

The other wavelet coefficients are of size 2^{-sj}; the liminf in (19) is never attained on those coefficients because we have:

$$\frac{\gamma d}{\alpha(d+(\gamma-s)p)} < s.$$

It follows that the Hölder exponent at the points of J_α is

$$\frac{\gamma d}{\alpha(d+(\gamma-s)p)}.$$

Using the trivial covering by the balls of radius $2^{-\alpha L}$ centered at the λ such that $c_\lambda^{(i)} \neq 0$, we obtain that the Hausdorff dimension of J_α does not exceed d/α. Since the balls centered at the same points and of radius 2^{-L} cover $[0,1]^d$, we are exactly in the position to apply Theorem 2 of [16] which asserts that $\mathcal{H}^{h_{d/\alpha}}(J_\alpha) > 0$ (so that the Hausdorff dimension of J_α is larger than d/α). Following Theorem 2.1 of [18], the set of points where a function in $B_p^{s,q} \cap C^\gamma$ has a Hölder exponent less than $\frac{\gamma d}{\alpha(d+(\gamma-s)p)}$ has a $\mathcal{H}^{h_{d/\alpha}}$ Hausdorff measure vanishing. It follows that the set of points where the Hölder exponent is exactly $\frac{\gamma d}{\alpha(d+(\gamma-s)p)}$ has the dimension d/α. Since the exponent of approximation belongs to $[1, \frac{d}{d+(\gamma-s)p}]$; the second point of Proposition 3 follows.

Furthermore, since every point satisfies (37) with $\alpha = 1$, for almost every x

$$h_F(x) = \frac{d\gamma}{d - sp + \gamma p};$$

the last point of Proposition 3 follows. □

In order to obtain a residual set, we follow the construction of Section 2.2. Using the same notations, we now pick for radius of the balls $B(g_n, r_n)$

$$r_n = \frac{1}{2n^{2/q}} 2^{-sn}$$

so that, at the scale $j = n$, the coefficients of a function of $B(g_n, r_n)$ is at least half the corresponding coefficient of F. We define now the residual set as:

$$A = \bigcap_{m\in\mathbf{N}} \bigcup_{n\geqslant m} B(g_n, r_n). \tag{39}$$

If a function f belongs to A, it belongs to an infinite number of balls $B(g_n, r_n)$. Denote by B_{n_l} this sequence of balls, and by K_α the set of points x satisfying the following property:

For an infinite number of values of j there exists k satisfying (35) and such that

$$\left| x - \frac{k}{2^j} \right| \leqslant \frac{1}{2^{\alpha L}}.$$

The same calculation as above yields $\frac{\gamma d}{\alpha(d+(\gamma-s)p)}$ as upper bound of the Hölder exponent of f at such a point $x \in K_\alpha$.

The derivation of the dimension of K_α is, as before, a direct consequence of Theorem 2 of [16] which yields that $\mathcal{H}^{h_{d/\alpha}}(K_\alpha) > 0$ (so that the Hausdorff dimension of K_α is larger than d/α). The set of points where a function in $B_p^{s,q} \cap C^\gamma$ has a Hölder exponent less than $\frac{\gamma d}{\alpha(d+(\gamma-s)p)}$ has a $\mathcal{H}^{h_{d/\alpha}}$ Hausdorff measure vanishing. It follows that the set of points where the Hölder exponent of f is exactly $\frac{\gamma d}{\alpha(d+(\gamma-s)p)}$ has the dimension d/α. Thus its spectrum of singularities is defined on $[\gamma, \frac{\gamma d}{d+(\gamma-s)p}]$ where:

$$d(H) = \frac{d + (\gamma - s)p}{\gamma} H.$$

Furthermore, the same argument as above shows that the Hölder exponent takes almost everywhere the value $\frac{d\gamma}{d-sp+\gamma p}$.

The adaptation of this proof in order to deal with the case of $L^{p,s} \cap C^\gamma$ is straightforward and left to the reader. Theorem 2 follows.

2.6. Generic validity of the multifractal formalism

The following propositions show the difference between the two cases depending whether $s_0 - d/p_0$ is positive or negative.

PROPOSITION 4. – *If $s_0 - d/p_0 > 0$, quasi-all function in the Besov space $B_{p_0}^{s_0,q_0}$ satisfy*

$$\eta(p) = \begin{cases} ps_0 & \text{if } p \leqslant p_0, \\ d + p\big(s_0 - \frac{d}{p_0}\big) & \text{if } p \geqslant p_0. \end{cases} \tag{40}$$

The second assertion of this proposition shows that the Besov embeddings are sharp for quasi-all functions (and, therefore, the Sobolev embeddings are also sharp for quasi-all functions): Functions of a Besov space strive to be as unsmooth as allowed by the Besov embeddings.

PROPOSITION 5. – *If $s_0 - d/p_0 < 0$, quasi-all functions in $B_{p_0}^{s_0,q_0} \cap C^\gamma$ satisfy*

$$\eta(p) = \begin{cases} ps_0 & \text{if } p \leqslant p_0, \\ (s_0 - \gamma)p_0 + \gamma p & \text{if } p \geqslant p_0. \end{cases} \tag{41}$$

Remarks. – In the case $s_0 - d/p_0 > 0$ the multifractal formalism yields for quasi-all functions:

$$\inf_{p>0}\big(d - \eta(p) + Hp\big) = \begin{cases} -\infty & \text{if } H < s_0 - d/p_0, \\ d - p_0 s_0 + Hp_0 & \text{if } s_0 - d/p_0 \leqslant H \leqslant s_0, \\ d & \text{if } H > s_0; \end{cases} \tag{42}$$

thus it yields correctly the increasing part of the spectrum (12).

In the case of $B_{p_0}^{s_0,q_0} \cap C^\gamma$, the 'usual' multifractal formalism (where the Legendre transform is taken for all ps positive) yields:

$$\inf_{p>0}\big(d - \eta(p) + Hp\big) = \begin{cases} -\infty & \text{if } H \leqslant \gamma, \\ d - s_0 p_0 + Hp_0 & \text{if } \gamma \leqslant H \leqslant s_0, \\ d & \text{if } H \geqslant s_0, \end{cases} \tag{43}$$

and we do not obtain the right spectrum given by Proposition 3. On the opposite, let us now restrict the range of p's on which the infimum in the Legendre transform is taken to the values for which a continous embedding holds; i.e. for the ps such that $\eta(p) > d$. It means that the infimum is taken for $p > p_c$ where:

$$p_c = \frac{d - (s_0 - \gamma)p_0}{\gamma},$$

p_c is larger than p_0 so that

$$\inf_{p>p_c}\big(d - \eta(p) + Hp\big) = \begin{cases} -\infty & \text{if } H < \gamma, \\ Hp_c = \frac{H(d-(s_0-\gamma)p_0)}{\gamma} & \text{if } H \geqslant \gamma \end{cases} \tag{44}$$

which yields the correct increasing part of the spectrum (13).

Proof of Proposition 4. – We start by determining the function $\eta(p)$ of the saturating function constructed in Section 2.1. There are $2^{dJ}(1-2^{-d})$ wavelet coefficients satisfying (20); thus

$$\begin{aligned}\sum_k \big|c_\lambda^{(i)} 2^{(s-\frac{d}{p})j}\big|^p &= \frac{(1-2^{-d})}{j^{ap}} \sum_{J=0}^{j} 2^{dJ}\big(2^{(\frac{d}{p_0}-s_0)j} 2^{\frac{-d}{p_0}J} 2^{(s-\frac{d}{p})j}\big)^p \\ &= \frac{(1-2^{-d})}{j^{ap}} \sum_{J=0}^{j} 2^{d(1-\frac{p}{p_0})J} 2^{p(\frac{d}{p_0}-\frac{d}{p}+s-s_0)j}.\end{aligned}$$

If $p \geqslant p_0$, this quantity is equivalent to $j^{-ap}2^{p(\frac{d}{p_0}-\frac{d}{p}+s-s_0)j}$ and $F \in B_p^{s,q}$ if and only if $s \leqslant s_0 + \frac{d}{p} - \frac{d}{p_0}$. It follows that, for $p \geqslant p_0$, $\eta(p) = d + p(s_0 - \frac{d}{p_0})$.

If $p < p_0$, $\sum_k |c_\lambda^{(i)} 2^{(s-\frac{d}{p})j}|^p$ is equivalent to $j^{-ap}2^{(s-s_0)j}$ and $F \in B_p^{s,q}$ if and only if $s \leqslant s_0$. It follows that $\eta(p) = sp_0$. Thus (40) holds for the saturating function F.

Let us now determine the function $\eta(p)$ for the elements of the residual set (32). First, we obtain a lower bound for $\eta(p)$. The Sobolev embeddings imply that, for $q \leqslant 1/p_0$, $s(q) \geqslant s_0 - d/p_0 + dq$ so that, for $p \geqslant p_0$, $\eta(p) \geqslant d + p(s_0 - d/p_0)$. Since $s(q)$ is increasing, for $q \geqslant 1/p_0$, $s(q) \geqslant s_0$, so that $\eta(p) \geqslant ps_0$ for $p \geqslant p_0$.

In order to obtain an upper bound, we remark that if f is an element of (32), keeping the same notations as in Section 2.2, at the scales $j = n_l$, f has each of its wavelet coefficients larger than half of the corresponding wavelet coefficients of F. Therefore, if $f \in B_q^{s,\infty}$, $F \in B_q^{s,\infty}$, and therefore the function $\eta_f(p)$ is smaller than $\eta_F(p)$. It follows that $\eta_F(p)$ is given by (40), and Proposition 4 holds. □

Proof of Proposition 5. – We consider now the case of $B_{p_0}^{s_0,q_0} \cap C^\gamma$ when $s_0 < d/p_0$. We start by determining the function $\eta(p)$ of the saturating function constructed in Section 2.5. There are 2^{dL} wavelet coefficients of size $j^{-2/q}2^{-\gamma j}$ where $L = [\frac{d+(\gamma-s_0)p_0}{d} j]$; and there are $2^{dj} - 2^{dL}$ wavelet coefficients of size $j^{-2/q}2^{-s_0 j}$; thus $\sum_k |c_\lambda^{(i)} 2^{(s-\frac{d}{p})j}|^p$ is equivalent to:

$$\begin{aligned} &j^{-2p/q}2^{(d+(\gamma-s_0)p_0)j}2^{-\gamma pj}2^{(sp-d)j} + j^{-2p/q}2^{dj}2^{-s_0pj}2^{(sp-d)j} \\ &\quad = j^{-2p/q}\bigl(2^{(\gamma(p_0-p)+sp-s_0p_0)j} + 2^{(s-s_0)pj}\bigr). \end{aligned}$$

It follows that $\eta(p) = \inf(s_0 p, (s_0-\gamma)p_0 + \gamma p)$, and Proposition 5 holds for F.

Let us now determine the function $\eta_f(p)$ for an element f of the set of generic functions A defined in (39). By interpolation, we have $s(q) \geqslant \gamma + qp_0(s_0 - \gamma)$ for $q \leqslant 1/p_0$, so that $\eta(p) \geqslant p_0(s_0 - \gamma) + p\gamma$ for for $p \geqslant p_0$.

Since $s(q)$ is increasing, for $q \geqslant 1/p_0$, $s(q) \geqslant s_0$; so that $\eta(p) \geqslant ps_0$ for $p \leqslant p_0$.

In order to obtain an upper bound, we consider a generic function f. Keeping the same notations as in Section 2.5, at the scales $j = n_l$, f has each of its wavelet coefficients larger than half of the corresponding wavelet coefficients of the saturating function F. Therefore, if $f \in B_q^{s,\infty}$, $F \in B_q^{s,\infty}$, and therefore the function $\eta_f(p)$ is smaller than $\eta_F(p)$. It follows that $\eta_F(p)$ is given by (41), and Proposition 5 holds. □

3. The Frisch–Parisi conjecture

In this section, we will check that V is a Baire space and prove Theorem 1. We first suppose that $\eta(p)$ is a strongly admissible function (in the sense of Definition 1); we consider the topological vector space V defined by:

$$V = \bigcap_{\varepsilon>0,\, 0<p<\infty} B_{p,\mathrm{loc}}^{(\eta(p)-\varepsilon)/p,\,p}. \tag{45}$$

Because of the Besov embeddings, see [27], V can be written as a countable intersection

$$V = \bigcap_{n\geqslant 1} B_n, \quad \text{where} \quad B_n = B_{p_n,\mathrm{loc}}^{(\eta(p_n)-\varepsilon_n)/p_n,\,p_n},$$

p_n being a dense sequence in $(0,+\infty)$ and $\varepsilon_n \to 0$. We can also make the additional assumption that

$$p_n \geqslant 1/\sqrt{n}. \tag{46}$$

Let $\varphi(x) \in \mathcal{D}(\mathbf{R}^d)$ such that $\varphi(x)=1$ if $x \in B(0,1)$ and $\varphi(x)=0$ outside $B(0,2)$ In the following we denote by $\|f\|_n$ the (pseudo)norm of a function $f\varphi(x/n)$ in B_n; Note that, though each B_n is either a Banach space (when $p_n \geqslant 1$) or a quasi-Banach space (when $p_n < 1$), V is not a pseudometric space because the constant involved in the definition of a pseudo-metric space is not bounded when $p \to 0$.

In V, a countable basis of neighbourhoods of the origin is given by the sets:

$$A_n = \left\{ f\colon \sup_{i=1,\dots,n} \|f\|_i \leqslant \frac{1}{n} \right\}.$$

Let E be a contable intersection of dense open subsets E_n of V, let $f \in V$, and let $\mathcal{D}$ be a neighbourhood of f in V. There exists N such that $\mathcal{D}$ contains the set

$$\mathcal{E} = \left\{ g\colon \sup_{i=1,\dots,N} \|f-g\|_i \leqslant \frac{1}{N} \right\}.$$

We want to find an element of $E = \bigcap_{n\in\mathbf{N}} E_n$ in $\mathcal{E}$.

Let $C_0 > 4N$. Since E_0 is dense in V, there exists $f_0 \in E_0$ such that:

$$\forall i = 1,\dots,N, \quad \|f - f_0\|_i \leqslant \frac{1}{C_0},$$

and since A_0 is open, there exists $n_0 \geqslant 4C_0$ such that:

$$\left\{ g\colon \sup_{i=1,\dots,n_0} \|g - f_0\|_i \leqslant \frac{1}{n_0} \right\}$$

is included in A_0.

Let $C_1 > 4n_0$. Since A_1 is dense in V, there exists $f_1 \in A_1$ such that:

$$\forall i = 1,\dots,n_0, \quad \|f_0 - f_1\|_i \leqslant \frac{1}{C_1},$$

and since A_1 is open, there exists $n_1 \geqslant 4C_1$ such that:

$$\left\{ g\colon \sup_{i=1,\dots,n_1} \|g - f_1\|_i \leqslant \frac{1}{n_1} \right\}$$

is included in A_1.

We continue this construction choosing a sequence C_n which grows fast enough so that the sequence f_m in V is a Cauchy sequence in each $B_{n,\mathrm{loc}}$ (which is complete) and has a limit in each A_n (the reader will easily check that, because of our choice (46), an exponential growth, with a large enough exponent, is sufficient). The sequence f_n thus converges in V to an element which belongs to $\bigcap_{n\in\mathbf{N}} A_n$ and also belongs to $\mathcal{E}$. Thus V is a Baire space.

We will now show that quasi-all functions of V satisfy formula (11). As above, we start by constructing a saturating function.

3.1. A saturating function adapted to $\eta(p)$

As in Section 2.1, the wavelet coefficients of the saturating function F will depend on the dyadic properties of the d-uple $k=(k_1,\dots,k_d)$, and for the same reasons, we can consider only the points of $(0,1)^d$ in the computation of the Hölder exponent. However, we have to take the extra care that the saturating function thus constructed:

- has wavelet coefficients small enough so that it belongs to all the Besov spaces that appear in (45),
- has wavelet coefficients large enough so that these inclusions cannot be improved (they are 'saturated').

Using the definition of J given in (20), let

$$a(j,k)=\inf_p\left(\frac{d(j-J)-\eta(p)j}{p}\right), \tag{47}$$

we define the wavelet coefficients of F by

$$c_\lambda^{(i)}=\frac{1}{j^a}2^{a(j,k)}, \tag{48}$$

with $a=\frac{2}{p}+1$ (of course, if the infimum in (47) is $-\infty$, we pick $c_\lambda^{(i)}=0$).

PROPOSITION 6. – *The saturating function F belongs to the space V defined in* (45)*, and*

$$\forall p>0,\quad \eta_F(p)=\eta(p). \tag{49}$$

Remark. – This proposition implies that the necessary conditions of admissibility required in Definition 1 are also sufficient, and thus that Proposition 1 holds.

Proof of Proposition 6. – Let p_0 be given and let $s_0=\eta(p_0)/p_0$. Since

$$a(j,k)\leqslant\frac{d(j-J)}{p_0}-s_0 j,$$

the coefficients (48) are smaller than (21), so that Proposition 2 implies that F belongs to $B_{p_0}^{\eta(p_0)/p_0,\infty}$; hence F belongs to the space V defined by (45) and $\eta_F(p)\geqslant\eta(p)$.

We now prove that $\eta_F(p)\leqslant\eta(p)$, i.e. that $\forall p$, $\forall\omega$, if $\omega>\eta(p)$, $F\notin B_p^{\omega/p,\infty}$. Instead of working with $\eta(p)$, we rather work with $s(q)=q\eta(1/q)$ because the admissibility conditions are more easily expressed in terms of $s(q)$. Let

$$\rho=d\left(1-\frac{J}{j}\right); \tag{50}$$

ρ takes discrete values between 0 and d with spacing d/j, and

$$a(j,k)=j\inf_q\bigl(\rho q-s(q)\bigr). \tag{51}$$

Thus

- if $0\leqslant\rho\leqslant s'(+\infty)$, the infimum is attained for $q=+\infty$ and is $-\infty$,
- if $s'(+\infty)\leqslant\rho\leqslant s'(0)$, the infimum is attained for a $q\in(0,\infty)$,

- if $\rho \geqslant s'(0)$, the infimum is attained for $q = 0$ and is $-s(0)$.

Let $q_0 \geqslant 0$ and $\beta > s(q_0)$. Let us check that $F \notin B^{\beta,\infty}_{1/q_0}$. Since $s(q)$ is concave and $0 \leqslant s'(q) \leqslant d$, we can find (for j large enough) a ρ of the type given by (50) and such that the line $\beta + \rho(q - q_0)$ lies above the graph of $s(q)$; thus

$$\exists \varepsilon > 0,\ \forall q > 0, \quad \beta + \rho(q - q_0) \geqslant s(q) + \varepsilon. \tag{52}$$

For this choice of ρ (i.e. for the corresponding choice of J given by (50))

$$a(j,k) \geqslant j(\varepsilon - \beta + \rho q_0) \geqslant j(\varepsilon - \beta) + d(j - J)q_0.$$

Therefore there are 2^{dJ} coefficients larger than $j^{-a}2^{j(\varepsilon-\beta)+d(j-J)q_0}$, so that

$$\begin{aligned}\sum_k \bigl|c_\lambda^{(i)} 2^{(\beta - dq_0)j}\bigr|^{1/q_0} &\geqslant j^{-a/q_0} 2^{dJ} \bigl(2^{j(\varepsilon-\beta)+d(j-J)q_0} 2^{(\beta-dq_0)j}\bigr)^{1/q_0} \\ &= j^{-a/q_0} 2^{\varepsilon j/q_0},\end{aligned}$$

which tends to infinity when $j \to +\infty$, so that $F \notin B^{s(q_0),\infty}_{1/q_0}$. Since this is true for any $q_0 > 0$, (49) follows. □

3.2. The Hölder exponent of *F*

PROPOSITION 7. – *The Hölder exponent of the saturating function F is*:

$$h_F(x_0) = \frac{1}{\alpha(x_0)} \inf_{\omega \geqslant \alpha(x_0)} \sup_q \bigl(\omega\bigl(s(q) - dq\bigr) + dq\bigr). \tag{53}$$

Proof of Proposition 7. – Let x_0 be fixed. We estimate as usual the order of magnitude of

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j} + |x_0 - \lambda|)}.$$

(A) *The wavelet coefficients such that λ is inside the cone of influence.*

In order to obtain a lower bound for the Hölder exponent, let us consider all the wavelet coefficients ‘inside the cone of influence of x_0’, i.e. such that j and k satisfy

$$\left|x_0 - \frac{k}{2^j}\right| \leqslant 2^{-j}. \tag{54}$$

Let $\varepsilon > 0$. For j and k given, we define J and K by $K/2^J = k/2^j$ (with $K \in \mathbf{Z}^d - (2\mathbf{Z})^d$). Since $\alpha(x_0)$ is the dyadic exponent at x_0, for J large enough,

$$\frac{1}{2^{(\alpha(x_0)+\varepsilon)J}} < \left|x_0 - \frac{K}{2^J}\right| = \left|x_0 - \frac{k}{2^j}\right|. \tag{55}$$

From the irreducibility of the fraction $K/2^J$, it follows that $j \geqslant J$, and from (55) and (54) it follows that:

$$j \leqslant \bigl(\alpha(x_0) + \varepsilon\bigr)J. \tag{56}$$

Furthermore, using again (54),

$$\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j}+|x_0-\lambda|)} = \frac{\log(|c_\lambda^{(i)}|)}{-j\log 2}\bigl(1+\mathrm{o}(1)\bigr) = \left(\sup_q dq\left(\frac{J}{j}-1\right)+s(q)\right)\bigl(1+\mathrm{o}(1)\bigr).$$

Because of (56), it is larger than

$$\left(\sup_q\left(dq\left(\frac{1}{\alpha(x_0)+\varepsilon}-1\right)+s(q)\right)\right)\bigl(1+\mathrm{o}(1)\bigr). \tag{57}$$

This holds for any $\varepsilon>0$, and the $\mathrm{o}(1)\to 0$ when $J\to+\infty$; it follows that the liminf taken on the wavelet coefficients inside the cone of influence is larger than:

$$\sup_q\left(dq\left(\frac{1}{\alpha(x_0)}-1\right)+s(q)\right),$$

which is larger than (53), because it corresponds to choosing $\omega=\alpha(x_0)$ in (53).

(B) *The wavelet coefficients such that λ is outside the cone of influence.*

Now, the coefficients $c_\lambda^{(i)}$ are such that

$$\bigl|k/2^j-x_0\bigr|>2^{-j}. \tag{58}$$

It follows that:

$$\begin{aligned}\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j}+|x_0-\lambda|)} &= \frac{\log(|c_\lambda^{(i)}|)}{\log(|x_0-\lambda|)}\bigl(1+\mathrm{o}(1)\bigr)\\ &= \frac{\bigl(\sup_q dq(J-j)+js(q)\bigr)\log 2}{-\log(|x_0-\lambda|)}\bigl(1+\mathrm{o}(1)\bigr).\end{aligned} \tag{59}$$

(1) *A lower bound when $j>\alpha(x_0)J$.*

We consider the wavelet coefficients such that $j>\alpha(x_0)J$. Because of (55), (59) is larger than

$$\left(\sup_q\left(\frac{dq(J-j)+js(q)}{J(\alpha(x_0)+\varepsilon)}\right)\right)\bigl(1+\mathrm{o}(1)\bigr)\geqslant\frac{1}{\alpha(x_0)+\varepsilon}\left(\sup_q\left(\frac{j}{J}\bigl(s(q)-dq\bigr)+dq\right)\right)\bigl(1+\mathrm{o}(1)\bigr).$$

Since ε can be chosen arbitrarily small, it is larger than

$$\inf_{j/J\geqslant\alpha(x_0)}\frac{1}{\alpha(x_0)}\sup_q\left(\frac{j}{J}\bigl(s(q)-dq\bigr)+dq\right). \tag{60}$$

(2) *A lower bound when $j\leqslant\alpha(x_0)J$.*

We consider the wavelet coefficients such that $j\leqslant\alpha(x_0)J$. Because of (58), (59) is larger than

$$\left(\sup_q\left(\frac{dq(J-j)+js(q)}{j}\right)\right)\bigl(1+\mathrm{o}(1)\bigr)=\left(\sup_q dq\left(\frac{J}{j}-1\right)+s(q)\right)\bigl(1+\mathrm{o}(1)\bigr)$$

which, since $j\leqslant\alpha(x_0)J$, is larger than:

$$\left(\sup_q\left(dq\left(\frac{1}{\alpha(x_0)}-1\right)+s(q)\right)\right)\bigl(1+\mathrm{o}(1)\bigr),$$

which is larger than

$$\inf_{j/J\geqslant\alpha(x_0)}\frac{1}{\alpha(x_0)}\sup_q\left(\frac{j}{J}\bigl(s(q)-dq\bigr)+dq\right), \tag{61}$$

since it corresponds to taking $j=\alpha(x_0)J$ in the infimum.

(3) *An upper bound for the Hölder exponent.*

There exists an infinite number of (K_n, J_n) such that:

$$\left|x_0-\frac{K_n}{2^{J_n}}\right|\leqslant\frac{1}{2^{(\alpha(x_0)-\varepsilon)J_n}}.$$

We pick j and k such that:

$$\frac{k}{2^j}=\frac{K_n}{2^{J_n}}\quad\text{and}\quad j\geqslant\bigl[\bigl(\alpha(x_0)+\varepsilon\bigr)J_n\bigr]+1.$$

This choice is possible because it implies that $j\geqslant J_n$ and because

$$\bigl|k2^{-j}-x_0\bigr|\geqslant\frac{1}{2^{(\alpha(x_0)+\varepsilon)J_n}}\geqslant 2^{-j}$$

so that λ is indeed outside the cone of influence.

Using (59), it follows that, for such a couple (j,k), the corresponding wavelet coefficients satisfy:

$$\begin{aligned}\frac{\log(|c_\lambda^{(i)}|)}{\log(2^{-j}+|x_0-\lambda|)}&\leqslant\left(\sup_q\left(\frac{dq(J_n-j)+js(q)}{J_n(\alpha(x_0)-\varepsilon)}\right)\right)\bigl(1+\mathrm{o}(1)\bigr)\\&\leqslant\frac{1}{\alpha(x_0)-\varepsilon}\left(\sup_q\left(\frac{j}{J_n}\bigl(s(q)-dq\bigr)+dq\right)\right)\bigl(1+\mathrm{o}(1)\bigr).\end{aligned}$$

Since ε can be chosen arbitrarily small, and j/J_N can be chosen arbitrarily close to any real number larger than $\alpha(x_0)$,

$$h_f(x_0)\leqslant\inf_{\omega\geqslant\alpha(x_0)}\frac{1}{\alpha(x_0)}\sup_q\bigl(\omega\bigl(s(q)-dq\bigr)+dq\bigr).$$

This ends the proof of Proposition 7. □

3.3. The spectrum of singularities of the saturating function

Let us now rewrite the Hölder exponent in a more convenient form. For that we rewrite (53) as:

$$h_f(x_0)=\frac{1}{\alpha(x_0)}\inf_{\omega\geqslant\alpha(x_0)}G(\omega), \tag{62}$$

where

$$G(\omega)=\omega\sup_{q>0}\left(s(q)-d\left(1-\frac{1}{\omega}\right)q\right). \tag{63}$$

Thus, if $\tilde{s}(h) = \sup_{q>0} s(q) - hq$ is the Legendre transform of $s(q)$,

$$G(\omega) = \omega\tilde{s}\left(d\left(1 - \frac{1}{\omega}\right)\right). \tag{64}$$

The following properties follow immediately from the definition of $\tilde{s}$ and the properties of s:
- $\tilde{s}(h) = +\infty$ for $h < s'(+\infty)$;
- $\tilde{s}(h) = s(0)$ for $h > s'(0)$;
- $\tilde{s}(h)$ is a convex decreasing function.

The following properties follow for G:
- $G(\omega) = +\infty$ for $\omega < \frac{d}{d-s'(+\infty)}$;
- $G(\omega) = \omega s(0)$ for $\omega \geqslant \frac{d}{d-s'(0)}$;

If G is twice differentiable, it follows from (64) that:

$$G''(\omega) = \frac{d^2}{\omega^3}\tilde{s}''\left(d\left(1 - \frac{1}{\omega}\right)\right)$$

so that G is convex. The case where G is not twice differentiable follows by a standard approximation argument.

Denote by ω_0, the value of ω for which the infimum of G is attained; then $G'(\omega_0) = 0$. Let us determine $G(\omega_0)$. Let $q(\omega)$ be the value of q where the supremum in (63) is attained. Since $\omega_0 \in (\frac{d}{d-s'(+\infty)}, \frac{d}{d-s'(0)})$, $q(\omega)$ is finite and non-zero in a neighbourhood of ω_0, and is obtained by annulating the derivative of the function $q \to \omega(s(q) - dq) + dq$, which yields

$$\omega\big(s'\big(q(\omega)\big) - d\big) + d = 0. \tag{65}$$

Therefore

$$G(\omega) = \omega\big(s\big(q(\omega)\big) - dq(\omega)\big) + dq(\omega).$$

The condition $G'(\omega_0) = 0$ thus becomes:

$$s\big(q(\omega_0)\big) - dq(\omega_0) + \omega_0\big(s'\big(q(\omega_0)\big) - d\big)q'(\omega_0) + dq'(\omega_0) = 0$$

which, using (65), implies that $s(q(\omega_0)) - dq(\omega_0) = 0$, so that, if q_c denotes $1/p_c$, $q(\omega_0) = q_c$, and therefore $G(\omega_0) = dq_c$. It follows that G is decreasing for $\omega \leqslant \omega_0$, and increasing for $\omega \geqslant \omega_0$.

Let us now sum up the previous results and deduce the spectrum of singularities of the saturating function F (using the fact that the Hausdorff dimension of the set of points with dyadic exponent α is exactly d/α). Let $H(\alpha) = \frac{1}{\alpha}\sup_{\omega\geqslant\alpha} G(\omega)$.
- If $\alpha \geqslant \frac{d}{d-s'(0)}$, $H(\alpha) = s(0)$ and the corresponding dimension is therefore $d(s(0)) = d - s'(0)$.
- If $1 \leqslant \alpha \leqslant \frac{d}{d-s'(q_c)}$, $H(\alpha) = \frac{1}{\alpha}G(\omega_0) = \frac{dq_c}{\alpha}$ and the corresponding dimension is therefore $d(H) = \frac{H}{q_c}$ for $H \in [q_c(d - s'(q_c)), dq_c]$.
- If $\frac{d}{d-s'(q_c)} \leqslant \alpha \leqslant \frac{d}{d-s'(0)}$,

$$H(\alpha) = \frac{1}{\alpha}G(\alpha) = \sup_q\left(s(q) - dq + \frac{dq}{\alpha}\right) \tag{66}$$

and the corresponding dimension is $d(H) = d/\alpha$. Thus

$$H = \sup_q s(q) - dq + d(H)q = \sup_p \left(\frac{\eta(p) - d + d(H)}{p} \right).$$

Therefore $\forall H, p$

$$H \geqslant \frac{\eta(p) - d + d(H)}{p}$$

so that $d(H) \leqslant Hp - \eta(p) + d$ with equality for one p, so that:

$$d(H) = \inf_p \bigl(Hp - \eta(p) + d\bigr).$$

Note that $H(\alpha)$ in (66) is a decreasing function of α. Thus $d(H)$ is an increasing function of H. If $\alpha = \frac{d}{d - s'(q)}$, we obtain $H(\alpha) = q_c(d - s'(q_c))$, and if $\alpha = \frac{d}{d - s'(0)}$, we obtain $H(\alpha) = s(0)$. Thus, in this case H takes values in $[s(0), q_c(d - s'(q_c))]$ where it is increasing.

Note also that, since $q(\omega)$ satisfies

$$s'(q(\omega)) = d\left(1 - \frac{1}{\omega}\right),$$

and since $s'(q)$ is decreasing, it follows that $q(\omega)$ decreases from q_c to 0 when ω increases from ω_0 to $\frac{d}{d - s'(0)}$. Since $q(\omega)$ is the point where the supremum is reached in (63), it follows that the range of qs can be restricted to $[0, q_c]$, and thus the range of ps to $[p_c, +\infty]$.

Finally, from the relationship $\eta(p) = ps(1/p)$ and from the definition of q_c, it follows that

$$\eta'(p_c) = q_c\bigl(d - s'(q_c)\bigr).$$

Let us now compute

$$\inf_{p \geqslant p_c} \bigl(Hp - \eta(p) + d\bigr) \tag{67}$$

for $H > \eta'(p_c)$. The infimum in (67) is attained for $p = p_c$ so that its value is

$$Hp_c - \eta(p_c) + d = Hp_c = \frac{H}{q_c}.$$

Note that, since $\alpha(x) = 1$ for almost every x, so that $h_F(x) = d/p_c$ for almost every x. Then we have the proposition:

PROPOSITION 8. – *The domain of definition of the spectrum of singularities of the saturating function F is the interval $[s(0), d/p_c]$ and, on this interval*

$$d(H) = \inf_{p \geqslant p_c} \bigl(Hp - \eta(p) + d\bigr). \tag{68}$$

Furthermore, $h_F(x) = d/p_c$ for almost every x.

We will use in the following a slightly different saturating function, which is obtained by adding $2.2^{-j \log j}$ to each wavelet coefficient $C_{j,k}$ of F. Since it corresponds to adding a C^∞ function, it affects neither the spectrum of singularities, nor the scaling function η_F.

3.4. Upper bounds for spectra

Let us recall that, if $\eta(p)$ is strongly admissible, and if p_c denotes the critical value for which $\eta(p) = d$, (10) yields

$$d(H) \leqslant \inf_{p \geqslant p_c} \big(pH + d - \eta(p)\big). \tag{69}$$

We can apply (14) for $\gamma = s(0)$ and for all $p \leqslant p_c$, which yields

$$d(H) \leqslant \inf_{p \leqslant p_c} \left(pH + \frac{H}{s(0)}\big(d - \eta(p)\big)\right). \tag{70}$$

Let us now prove that, if (69) holds, then (70) holds, so that we can only keep (69) as a condition for the spectrum.

For a given H, we consider the function

$$pH + \frac{H}{s(0)}\big(d - \eta(p)\big). \tag{71}$$

We write it as usual as a function of $q = 1/p$; since $s(q) = q\eta(p)$, (71) becomes

$$g_H(q) = \frac{H}{q} + \frac{H}{s(0)}\left(d - \frac{s(q)}{q}\right);$$

thus

$$g_H'(q) = \frac{H}{q^2 s(0)}\big(s(q) - s(0) - qs'(q)\big)$$

which is positive because of the concavity of $s(q)$. It follows that $g_H(q)$ is minimal for $q = q_c$, and thus that the infimum in (70) is attained for $p = p_c$, which corresponds to the case $p = p_c$ in (69). Therefore, if (69) holds, (70) holds a fortiori. Since $H \geqslant s(0)$, it follows that all the upper bounds for spectra that we have become equalities for the saturating function when $H \in [0, dq_c]$.

3.5. The residual set

The space V is separable, since finite linear combinations of wavelets with rational coefficients are clearly dense. We can therefore pick a sequence f_n which is dense in V. We denote by g_n the following modification of f_n:

- If $j < n$ the wavelet coefficients of g_n are the same as those of f_n.
- If $j \geqslant n$ the wavelet coefficients $c_\lambda^{(i)}$ of g_n are the same as those of the saturating function F constructed in the previous subsection.

It clearly follows from our choice of the wavelet coefficients of the saturating function F that the g_n are also dense in V. The G_δ dense set that we will consider is

$$A = \bigcap_{m \in \mathbf{N}} \bigcup_{n \geqslant m} B(g_n, r_n), \tag{72}$$

where $r_n = 2^{-n \log n}$.

A is clearly a countable intersection of dense open sets. We have chosen r_n small enough so that, at the scale $j = n$ each wavelet coefficients of the functions in $B(g_n, r_n)$ is close to the corresponding coefficient of g_n; indeed, the smallest coefficient of the saturation function F at the scale $j = n$ is larger than $2^{-n\log n}$, and r_n is chosen so that the wavelet coefficients of f differ from those of f by (much) less that $\frac{1}{2}.2^{-n\log n}$.

If a function f belongs to the residual set A, it belongs to an infinite number of balls $B(g_n, r_n)$. Denote by B_{n_l} this sequence of balls. Thus, at the scales $j = n_l$, the wavelet coefficients of f are 'close' to those of the saturating function F. We denote by F_α the set of points x such that

$$\exists K\colon \left| x - \frac{K}{2^{[n_l/\alpha]}} \right| \leqslant \frac{1}{2^{n_l}}$$

for an infinite number of values of l. This set can also be defined as

$$\limsup_{l\to\infty} \bigcup_{m\geqslant l,K} \frac{K}{2^{[n_l/\alpha]}} + \left[-\frac{1}{2^{n_l}}, \frac{1}{2^{n_l}} \right]^d.$$

The Hölder exponent of f at such a point $x \in F_\alpha$ is estimated exactly as in Section 2.2 and we obtain that the Hölder exponent at this point is smaller than $H(\alpha)$.

In order to compute the dimension of F_α, we apply apply again Theorem 2 of [16] which asserts that $\mathcal{H}^{h_{d/\alpha}}(F_\alpha) > 0$ (so that the Hausdorff dimension of F_α is larger than d/α). Following Theorem 2.1 of [18], the set of points where a function in $B_p^{s,q}$ has a Hölder exponent less than $H(\alpha)$ has a $\mathcal{H}^{h_{d/\alpha}}$ Hausdorff measure vanishing. It follows that the set of points where the Hölder exponent is exactly $H(\alpha)$ has the dimension d/α, and (11) holds. Furthermore, since every x belongs to F_1, it follows that the Hölder exponent is $h_f(x) = d/p_c$ for almost every x, and the first point of Theorem 1 holds.

As regards the second part of Theorem 1, we note that, since $\eta(p)$ is not strongly admissible, $s(q) \leqslant dq$; therefore, we can pick for saturating function the function defined in Section 2.4. The argument developed in Section 2.4 applies here also without any change and yields second part of Theorem 1.

Concluding remarks

One may wonder why the Frisch–Parisi formula is generically wrong as stated unsually, i.e. when the infimum in (5) is taken on all $p > 0$. By inspecting the proof of the determination of the Hölder exponent of the saturating functions when $H > \eta'(p_c)$, the reader will check that the Hölder exponent is not determined by the wavelet coefficients inside the cone of influence (as is the case for $H \leqslant \eta'(p_c)$) but rather by wavelet coefficients in 'tangential domains' $2^{-j} \sim |x_0 - j2^{-j}|^{1+\beta}$ for a $\beta > 0$. This behavior is characteristic of *oscillating singularities*, a typical example of which is supplied by the functions

$$|x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta} \right).$$

(Note that this notion is a slight variant of the *chirps* studied by Yves Meyer in [18] and [26].) Such behaviors are studied in [2] and [15] where it is shown why the multifractal formalism fails for functions which include such local oscillatory behaviors. (Indeed, the heuristic argument usually advocated to justify the multifractal formalism makes the implicit assumption that the Hölder exponent at a point is given by the rate of decay of the wavelet transform inside the cone of influence at this point.) It is therefore not surprising that, if such Hölder singularities appear for $H \geqslant \eta'(p_c)$, the usual multifractal formalism fails in this range of Hölder exponents.

Acknowledgement

The author is very grateful to Jean-Pierre Kahane and Yves Meyer for many comments on this text.

Note added on proofs

Since this paper was submitted, Yves Meyer and the author extended formula (10) and Theorem 2 to the critical Besov spaces (where $s = d/p$), see "On the pointwise regularity of functions in the critical Besov spaces", preprint.

REFERENCES

[1] M. Abouda, Singularités et analyse 2-microlocale des mesures et distributions fractales, Thèse de l'Université Paris 9 Dauphine, 1998.

[2] A. Arneodo, E. Bacry, S. Jaffard and J.-F. Muzy, Singularity spectrum of multifractal functions involving oscillating singularities, *J. Four. Anal. Appl.* 4 (2) (1998) 159–174.

[3] A. Arneodo, E. Bacry and J.-F. Muzy, The thermodynamics of fractals revisited with wavelets, *Physica A* 213 (1995) 232–275.

[4] S. Banach, Über die Baire'sche Kategorie gewisser Funktionenmengen, *Studia Math.* 3 (1931) 174–179.

[5] J. Barral, Moments, continuité et analyse multifractale des martingales de Mandelbrot, *Probab. Theory Related Fields* 113 (1999) 535–570.

[6] Z. Buczolich and J. Nagy, Hölder spectrum of typical monotone continuous functions, Preprint, Eötvös Loránd University, Budapest, 1999.

[7] G. Brown, G. Michon and J. Peyrière, On the multifractal analysis of measures, *J. Stat. Phys.* 66 (1992) 775–790.

[8] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* 41 (1988) 909–996.

[9] I. Daubechies and J. Lagarias, On the thermodynamic formalism for multifractal functions; in: M. Aizenman and H. Araki (Eds.), *The State of Matter* (dedicated to Elliott Lieb), World Scientific, Singapore, 1994, pp. 213–265.

[10] K. Falconer, *Fractal Geometry*, John Wiley, 1990.

[11] U. Frisch and G. Parisi, On the singularity structure of fully developed turbulence; appendix to Fully developped turbulence and intermittency, by U. Frisch, in: *Proc. Int. Summer School Phys. Enrico Fermi*, North-Holland, 1985, pp. 84–88.

[12] Y. Gagne, Etude expérimentale de l'intermittence et des singularités dans le plan complexe en turbulence pleinement développée, Thèse de l'Université de Grenoble, 1987.

[13] S. Jaffard, *Pointwise Smoothness, Two-Microlocalisation and Wavelet Coefficients*, Publicacions Matemàtiques, Vol. 34, 1990.

[14] S. Jaffard, Multifractal formalism for functions. Part I: Results valid for all functions, *SIAM J. Math. Anal.* 28 (1997) 944–970.

[15] S. Jaffard, Oscillation spaces: Properties and applications to fractal and multifractal functions, *J. Math. Phys.* 39 (8) (1998) 4129–4141.

[16] S. Jaffard, On lacunary wavelet series, *Ann. Appl. Probab.*, to appear.

[17] S. Jaffard, Generalization of oscillation spaces to negative ps: How to recover the right part of a spectrum of singularities?, Preprint.

[18] S. Jaffard and Y. Meyer, Wavelet methods for pointwise regularity and local oscillations of functions, *Mem. Amer. Math. Soc.* 123 (587) (1996).

[19] V. Jarnik, Über die diefferentierbarkeit stetiger Funktionen, *Fund. Math.* 21 (1933) 48–58.

[20] J.-P. Kahane and J. Peyrière, Sur certaines martingales de Benoit Mandelbrot, *Adv. Math.* 22 (1979) 131–145.

[21] P.-G. LEMARIÉ and Y. MEYER, Ondelettes et bases hilbertiennes, *Rev. Mat. Iberoamericana* 1 (1986) 1–17.
[22] B. MANDELBROT, Intermittent turbulence in selfsimilar cascades: divergence of high moments and dimension of the carrier, *J. Fluid Mech.* 62 (1974) 331.
[23] B. MANDELBROT, Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire, *C. R. Acad. Sci. Paris* 278 (1974) 289–292.
[24] S. MAZURKIEWICZ, Sur les fonctions non dérivables, *Studia Math.* 3 (1931) 92–94.
[25] Y. MEYER, *Ondelettes et Opérateurs*, Vol. 1, Hermann, Paris, 1990.
[26] Y. MEYER, *Wavelets, Vibrations and Scalings*, CRM Ser. AMS, Vol. 9, Presses de l'Université de Montréal, 1998.
[27] J. PEETRE, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser. I, 1976.
[28] S. SAKS, On the functions of Besicovitch in the space of continuous functions, *Fund. Math.* 19 (1932) 211–219.
[29] T. ZAMFIRESCU, Most monotone functions are singular, *Amer. Math. Monthly* 88 (1981) 47–49.
[30] T. ZAMFIRESCU, Typical monotone continuous functions, *Arch. Math.* 42 (1984) 151–156.