Rigourous math course with physics motivations. Parts with physics non-examinable and will be clearly marked. ⊛ Plan of course:

Topic 0: Preliminaries (first 3 classes, 4^{th} class overview)

Topic 1: Nodal sets of Laplace eigenfns. (~2 classes)

Topic 2: Geometric properties of rough fns. (some probability) (3 classes)

Topic 3: Multifractality & Energy Conservation results. (4 classes)

Advanced topics: Anomalous energy dissipation, ~~Vanishing visco~~ Turbulence in 2D, ...

⊛ What is 3D Turbulence? Simply random or universal & structured? ← Ref: Frisch Turbulence

0.1 Incompressible Fluids: Let us fix $(x,t) \in \mathbb{T}^3 \times \mathbb{R}$, $u: \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3$, $p: \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$

$$\nu = 0\ (\text{Euler}) \quad (\text{N-S}) \begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u, & \nu \geq 0 \\ \operatorname{div} u = 0 \end{cases}$$

Summation notation: $\nabla = \partial_i$, $(u \cdot \nabla u) = u_i \partial_i u_j = \partial_i (u_i u_j) = \operatorname{div}(u \otimes u)$

p can be recovered from u: $\operatorname{div}\operatorname{div}(u \otimes u) + \Delta p = 0$

⇒ Momentum is conserved.

Energy identity: (Let $u \in C^1_{t,x}$)

$$\partial_t \frac{|u|^2}{2} + (u \cdot \nabla)\frac{|u|^2}{2} + \operatorname{div}(pu) = \nu \Delta \frac{|u|^2}{2} - \nu |\nabla u|^2$$

$$\Rightarrow \frac{d}{dt} \frac{\|u\|_{L^2}^2}{2} = -\nu \int_{\mathbb{T}^3} |\nabla u|^2 \Rightarrow \|u\|_2^2(T) = \|u\|_2^2(0) - \int_0^T \int_{\mathbb{T}^3} 2\nu |\nabla u|^2$$

$\nu = 0 \Rightarrow$ Energy is conserved.

PHYSICS: Anomalous dissipation (0^{th} Law of Turbulence)

We observe $\liminf_{\nu \to 0} \nu \int_0^T \int_{\mathbb{T}^3} |\nabla u^\nu|^2 \geq c_0 > 0$. How do we describe the limiting soln. of the Euler eqns? By above it cannot be a C^1 soln.

Weak derivatives & Weak Solns.

Defn.: We say $g \in L^1(\Omega)$ is the weak derivative of $f \in L^1(\Omega)$ if $\forall \varphi \in C_c^\infty(\Omega)$

$$\int f \partial_i \varphi = -\int g \varphi.$$

Defn.: $u \in L^2(\mathbb{T}^3 \times \mathbb{R})$ is a weak soln. of Euler/N-S if $\forall \varphi \in C_c^\infty(\mathbb{T}^3 \times \mathbb{R})$ we have

$$-\iint u \partial_t \varphi + u \otimes u \cdot \nabla \varphi + p \nabla \varphi = \nu \iint u \Delta \varphi, \quad \iint u \cdot \nabla \varphi = 0$$

Eg: (Shear flows): $u(t, x_1, x_2, x_3) = \tilde{u}(x_1, x_2)\vec{e}_3$, $p \equiv 0$ for any $\tilde{u} \in L^2(\mathbb{T}^2)$ is a weak soln. of Euler.

The weak solns. as defined need not satisfy the energy identity!

Thm (Paradox of Scheffer): $\exists (u,p)$ weak soln of (Euler) which is compactly supported in spacetime.

No anomalous dissipation in 2D: Taking curl of N-S gives $\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega \\ u = K * \omega \end{cases}$

~~u incompressible~~ $\Rightarrow \frac{d}{dt} \|\omega\|_{L^2}^2 = -\nu \int |\nabla \omega|^2 \leq 0$

$\Rightarrow \|\omega\|_{L^2} \sim \|u\|_{\dot{H}^1} \leq C$ indep of ν $\Rightarrow \nu \|\nabla u\|_{L^2}^2 \to 0$.

⚡ Doesn't work in 3D due to "vortex stretching" $\omega \cdot \nabla u$

$\Omega = \mathbb{R}^n$ or $\mathbb{T}^n$.

0.2. Function Spaces: How do we measure regularity of fns. with less than 1 derivative.

Defn.: (Hölder spaces) For $0<\alpha<1$, we define $C^\alpha(\Omega)$ to be the set of all fns $f:\Omega\to\mathbb{R}$ s.t.

$$[f]_\alpha := \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < +\infty.$$

With the norm $\|f\|_\alpha := \|f\|_{C^0} + [f]_\alpha$ it is a Banach space.

Eg: $C^\alpha(\Omega)$ is NOT the closure of $C^\infty(\Omega)$ under the $\|\cdot\|_\alpha$ norm.

Defn.: (Sobolev spaces) For $p=2$, we define $H^\beta = W^{\beta,2}$ to be fns. $f:\Omega\to\mathbb{R}$ s.t.

$$\cancel{[f]_{H^\beta}}\ \|f\|_{H^\beta}^2 := \int (1+|\xi|^2)^\beta |\hat f(\xi)|^2\, d\xi < +\infty.$$

This is a Hilbert space: the inner product $\langle f,g\rangle_{H^\beta} = \int f\bar g\, dx + \int |\xi|^{2\beta} f(\xi)\bar g(-\xi)\, d\xi$ gives an equivalent norm.

$p\neq 2$ is more complicated, can be done by:
- Littlewood – Payley Theory (c.f. Alhinac – Gerard)
- Gagliardo Seminorm
- Wavelets

Defn: For $0<\alpha<1$, $\infty>p\geq 1$, $\|f\|_{\alpha,p} := \left(\iint \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha}}\, dx\,dy\right)^{1/p}$

$W^{\alpha,p} = \{f\in L^1 : \|f\|_{\alpha,p} < +\infty\}$.

PHYSICS: Experiments / numerics measure $\langle |u(x+\ell)-u(x)|^p\rangle \underset{\text{ergodic hypothesis}}{=} \|u(\cdot+\ell)-u(\cdot)\|_{L^p}^p$
"ensemble average"

Kolmogorov K41 theory:

$$\langle|\delta u(\ell)|^p\rangle \sim C|\ell|^{\zeta_p}, \quad \zeta_p = \frac{p}{3}.$$ Moreover, for $p=3$,

$$\langle|\delta u(\ell)|^3\rangle \sim c_\varepsilon |\ell|$$

Rmk: A rigorous version of such laws have been established. We'll look at this towards the end of the course.

The above motivates us to consider spaces so that $\|\delta u(\ell)\|_{L^p}^p \leq C|\ell|^{\alpha p}$. We get

Defn.: (Besov spaces) $$[f]_{B^\alpha_{p,q}} := \left(\int_0^\infty \left(\frac{\sup_{|\ell|\leq h}\|f(x+\ell)-f(x)\|_{L^p}}{h^\alpha}\right)^q \frac{dh}{h}\right)^{1/q}.$$

Eg.: $B^\alpha_{\infty,\infty} = C^\alpha$, $B^\alpha_{p,p} = W^{\alpha,p}$.

(*) Thm (Sobolev / Besov embedding): Let $0\leq\alpha<\beta<1$, $1\leq r\leq\infty$, $1\leq p<q\leq\infty$. If: $\frac{1}{p}-\frac{\beta}{d} = \frac{1}{q}-\frac{\alpha}{d}$

$\Rightarrow B^\beta_{p,r} \subseteq B^\alpha_{q,r}$

(*) Rmk: K41 theory suggests turbulent solns $u\in B^{1/3}_{3,\infty}$.

Rmk: We only care about the local picture, so we assume Ω is cpt or replace B_α with $B_{loc\,\beta}$

Thm (Interpolation): $1\leq p_0,q_0,p_1,q_1\leq\infty$, $0<\alpha,\beta<1$, $f\in B^\alpha_{p_0,q_0}\cap B^\beta_{p_1,q_1}$

$\Rightarrow$ for $0<\theta<1$, $\frac{1}{p} = \frac{\theta}{p_0}+\frac{1-\theta}{p_1}$, $\frac{1}{q} = \frac{\theta}{q_0}+\frac{(1-\theta)}{q_1}$, $\gamma = \theta\alpha+(1-\theta)\beta$, $f\in B^\gamma_{p,q}$.

Defn. (Structure function exponents): Given a function $f:\Omega\to R$, we define its p^{th}-order structure fn. exponent $\zeta_p(u) := p\cdot\sup\{\beta : u\in W^{\beta,p}\}$.

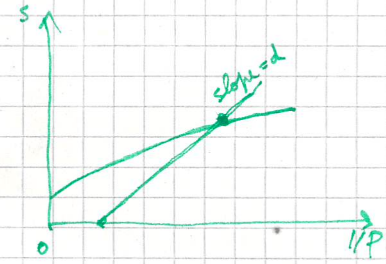
Defn. (Besov domain): The Besov domain of a fn. f $= \{(p,s) : f\in W^{s,\frac{1}{p}}\}$.

Interpolation $\Rightarrow$ Besov domain (f) is convex

$(q,s)\in B(f) \Rightarrow (t, d(t-q)+s)\in B(f)$ $\quad t\in(0,q)$

$(q,s)\in B(f) \Rightarrow (t,s)\in B(f)$ $\quad t\in(q,1)$

$\partial B(f)$ is given by $(q,s(q))$ (*) $\quad s(q)$ concave, increasing, $0\leq s'(q)\leq d$



Thm (Jaffard): Any convex fn s satisfying (*) gives the Besov domain of a function f.