0.3 Measures & Dimension

Defn.: An outer measure on X is a fn $\mu: 2^X \to [0,\infty]$ s.t.

$\circ\ \mu(\emptyset) = 0$, $\circ$ if $A \subseteq \bigcup_{i=1}^{\infty} B_i$, then $\mu(A) \leq \sum_i \mu(B_i)$

Defn. (Caratheodory): A set $A \subseteq X$ is μ-measurable if for every $Y \subseteq X$

$$\mu(Y) = \mu(Y \cap A) + \mu(Y \setminus A).$$

Thm. (Caratheodory): The collection of μ-measurable sets forms a σ-algebra.

Defn.: An (outer) measure μ is Borel regular if all Borel sets are μ-measurable and for any $A \subseteq X$ measurable there exists a Borel $B \supseteq A$ s.t. $\mu(A) = \mu(B)$.

Thm. (Caratheodory): If (X,d) a metric space, μ outer measure on X satisfys $\mu(A\cup B) = \mu(A)+\mu(B)$ if $d(A,B)>0$ then all Borel sets are μ-measurable.

Pf: Enough to show closed sets are measurable. WTS. for C closed set $\forall Y$, $\mu(Y) = \mu(Y\setminus C) + \mu(Y\cap C)$. Let $C_k = \{x : d(x,C) < \frac{1}{k}\}$

Then $d(Y\setminus C_k, Y\cap C) > 0$, so $\mu(Y) \geq \mu(Y\setminus C_k \cup Y\cap C) = \mu(Y\setminus C_k) + \mu(Y\cap C)$. ETS. $\mu(Y\setminus C_k) \to \mu(Y\setminus C)$.

$Y\setminus C = Y\setminus C_k \cup \bigcup_{j=k}^{\infty} R_j$. Show $\sum_{j=1}^{\infty} \mu(R_j) < \infty$. □

For Y any set and μ an outer measure, we define the outer measure $\mu \llcorner Y(A) := \mu(Y\cap A)$.

Examples: 1. Lebesgue measure. $X = \mathbb{R}^n$. Consider $I_a = [a_1,b_1]\times\dots\times[a_n,b_n]$ and define $\tilde{\mu}(I_a) = \prod(b_i - a_i)$

Now define the outer measure $\mathcal{L}^n(A) = \inf_{A\subseteq \bigcup I_i} \sum \tilde{\mu}(I_i)$.

$\mathcal{L}^n$ is Borel regular, in fact, it is regular. $\forall A$ measurable $\forall\varepsilon>0$ $\exists\, O \supseteq A$ open st $\mu(O) \leq \mu(A)+\varepsilon$.

The Lebesgue Differentiation Thm $\Rightarrow$ $\forall A$ st $\mu(A)>0$, $\forall \varepsilon>0$, $\exists$ a ball $B = B(x,r)$ st $(1-\varepsilon)\mu(B) \leq \mu(A\cap B)$.

Cool Fact: A configuration of m pts is an equivalence class of m pts $(p_1,\dots,p_m)$ up to dilations and translations. For any configuration C and every $A\subseteq\mathbb{R}^n$ st. $\mu(A)>0$, $\exists\, p_1,\dots,p_m \in$ ~~supp~~ A s.t. $(p_1,\dots,p_m) \in C$.

2. Hausdorff measure. $X = \mathbb{R}^n$. For any set C, we let $d(C) = \sup\{|x-y| : x,y\in C\}$ be its diam.

~~Let $A \subseteq \mathbb{R}^n$ be Borel~~

Define the outer measure $\mathcal{H}^s_\delta(A) = \inf_{A\subseteq\bigcup_{i=1}^{\infty} C_i} \left\{ \sum_{i=1}^{\infty} \omega_s \left(\frac{d(C_i)}{2}\right)^s : A \subseteq \bigcup_{i=1}^{\infty} C_i,\ d(C_i)\leq\delta \right\}$.

Note $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s_{\delta'}(A)$ for $0<\delta'<\delta$, so we can define $\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A)$.

$\mathcal{H}^s$ is Borel regular while $\mathcal{H}^s_\delta$ is NOT!

Thm: $\mathcal{H}^n = \mathcal{H}^n_\infty = \mathcal{L}^n$.

Cantor middle thirds set: $E_0 = [0,1]$, $E_1 = E_0 \setminus (\frac{1}{3},\frac{2}{3})$, $E_2 = E_1 \setminus \left((\frac{1}{9},\frac{2}{9})\cup(\frac{7}{9},\frac{8}{9})\right), \dots$ $C = \bigcap_i E_i$

E_i is covered by 2^i intervals of length 3^{-i}, So $C \subseteq E_i = \bigcup_{j=1}^{2^i} I_{i,j}$. So for $s = \left|\frac{\log 2}{\log 3}\right|$

$\mathcal{H}^s_\delta(C) \leq \sum_{j=1}^{2^i} (3^{-i})^s \leq 2^i\cdot 2^{-i} = 1$ for $3^{-i}<\delta$

Defn (Hausdorff dimension): For $A\subseteq\mathbb{R}^n$

$\dim_H(A) = \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}$.

So we have $\dim_H(C) \leq \left|\frac{\log(2)}{\log(3)}\right|$. What about the lower bound?

Non-example: Minkowski content.

Defn: For $A\subseteq\mathbb{R}^n$, $B(A,\delta) = \{x\in\mathbb{R}^n ; \exists y\in A \text{ st } d(x,y)<\delta\}$

Upper s-Minkowski content $\overline{\mathrm{Mink}}^s(A) = \limsup_{\delta\to 0} \frac{\mathcal{L}^n(B(A,\delta))}{\omega_{n-s}\,\delta^{n-s}}$

Lower s-Minkowski content $\underline{\mathrm{Mink}}^s(A) = \liminf_{\delta\to 0}$ —"—.

Can define upper and lower Minkowski dimension similar to Hausdorff dim.

Eg: $\dim_H(\mathbb{Q}) = 0$ while $\dim_M(\mathbb{Q}) = 1$.

Eg: For a compact set A, $[\chi_A]_{s,1} = \infty \Rightarrow \overline{\mathrm{Mink}}^{n-s}(\partial A) = \infty$.