0.4 Frostman's Lemma

Thm (F.L): Let $0 \le s \le n$, $A \subseteq \mathbb{R}^n$ be ~~Borel~~ compact. Then $\mathcal{H}^s(A) > 0$ if and only if $\exists \mu \neq 0$ a Borel measure s.t. $\operatorname{supp}(\mu) \subseteq A$ $[\mu \in \mathcal{M}(A)]$ s.t. $\mu(B(x,r)) < r^s$ $\forall x \in \mathbb{R}^n, r > 0$.

Pf.: $\Leftarrow$. Let $A \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j)$ be a covering by balls. Now

$\sum d(B(x_j,r_j))^s \sim \sum r_j^s > \sum \mu(B(x_j,r_j)) \geq \mu(A) > 0$. Since the covering was arbitrary, $\mathcal{H}^s(A) > 0$.

$\Rightarrow$. ASSUME $\mathcal{H}^s(A) > 0 \Rightarrow \exists c$ st. $\mathcal{H}^s_\infty(A) \geq c > 0$. We first construct a measure μ_k which will be "μ resolved to scales 2^{-k}".

Let $\mathbb{R}^n = \bigsqcup_j C_{k,j}$ be the partition into cubes of sidelength 2^{-k} with sides parallel to the coordinate axes. Now define $f_{k,k}(x) = \begin{cases} d(C_{k,j})^s / \mathcal{L}^n(C_{k,j}) & \text{if } x \in C_{k,j} \text{ and } C_{k,j} \cap A \neq \emptyset. \\ 0 & \text{otherwise} \end{cases}$

Let $\mu_{k,k} = f_{k,k}\mathcal{L}^n$. Define inductively $f_{k,i}(x) = \begin{cases} f_{k,i-1}(x)\, d(C_{i,j})^s / \mu_{k,i-1}(C_{i,j}) & \text{if } x \in C_{ij} \text{ and } \mu_{k,i-1}(C_{ij}) > d(C_{ij})^s \\ f_{k,i-1}(x) & \text{otherwise} \end{cases}$

Observe $f_{k,0} \geq f_{k,k-1} \geq f_{k,k-2} \geq \dots$. So $\mu_{k,i} \to \mu_k$. Also $\mu_{k}(\mathbb{R}^n) \geq \mathcal{H}^s_\infty(A) > 0$

As $\mu_k(B(x,r)) \lesssim r^s$, $\mu_k \to \mu$ weakly and μ does the job. $\square$

Application:

Defn.: The s-energy of a Borel measure μ is $I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y)$.

Note, μ has compact support $\Rightarrow$ for $0 < t < s$ $I_s(\mu) < \infty \Rightarrow I_t(\mu) < \infty$.

Thm: Let $A \subseteq \mathbb{R}^n$ be ~~Borel~~ compact, then
$$\dim_{\mathcal{H}}(A) = \sup\{s : \exists \mu \in \mathcal{M}(A) \text{ with } I_s(\mu) < \infty\}.$$

Pf: Consider $I_s(\mu,x) = \int |x-y|^{-s}\, d\mu(y) \,\text{"}{=}\text{"}\, \int_0^\infty r^{-s} \frac{d}{dr}\mu(B(x,r))\, dr \,\text{"}{=}\text{"}\, s\int_0^\infty r^{-(s+1)} \mu(B(x,r))\, dr$

Let $\dim_{\mathcal{H}}(A) = s$, let μ_A be the Frostman measure.

Thus $I_{s-\varepsilon}(\mu_A, x) \leq (s-\varepsilon)\int_0^{N} r^{-1+\varepsilon}\, dr + (s-\varepsilon)\int_N^\infty r^{-(s+1)} C\, dr < \infty$

$\Rightarrow I_{s-\varepsilon}(\mu_A) < \infty$.

Now suppose $\exists 0 \le s \le n$ st $\exists \mu \in \mathcal{M}(A)$ s.t $I_\mu(A) < \infty \Rightarrow \int |x-y|^{-s} d\mu(y) < \infty$ for μ-a.e. x

$\Rightarrow \int_0^\infty r^{-(s+1)}\mu(B(x,r))\, dr < \infty$ for μ-a.e. x

Thus $\exists M > 0$ s.t $\mu(\underbrace{\{x : I(\mu,x) < M\}}_{=: B}) > 0$. Consider $\mu \llcorner B$. Clearly $\mu \llcorner B \in \mathcal{M}(A)$, $\mu \llcorner B \neq 0$,

$\mu \llcorner B(B(x,r)) \lesssim r^s$.

So, by Frostman's Lemma, $\mathcal{H}^s(A) > 0 \Rightarrow \dim_{\mathcal{H}}(A) \geq s$. $\square$

Question: Define $f(x) = \sum_{n=0}^\infty a^n \cos(\pi b^n x)$. For $ab > 1$, $0 < a < 1$ this is continuous but nowhere differentiable. What is the Hausdorff dim of the graph of f? (~~I think this is still open~~ Solved only in 2016)

Ref to weak convergence:

Mattila "Fourier Analysis and Hausdorff Dimension"

0.5 Wavelets

$X = \mathbb{R}^n$.

Idea: Want a basis of $L^2(\mathbb{R}^n)$ to localize both in frequency and physical space.

Eg 1 (the Haar basis): Let $\Psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, $\varphi = \chi_{[0,1)}$,

the set $\{2^{j/2}\Psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Let $\Psi^0 = \varphi$, $\Psi^1 = \Psi$, for $x \in \mathbb{R}^n$ $\Psi^{(e_1,\dots,e_n)}(x) = \Psi^{e_1}(x_1)\cdots\Psi^{e_n}(x_n)$ $e_i \in \{0,1\}$

then $\{\underbrace{2^{nj/2}\Psi^{(e_1,\dots,e_n)}(2^j x - k)}_{\Psi^{(e)}_{j,k}} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, (e_1,\dots,e_n) \in \{0,1\}^n \setminus \{(0,\dots,0)\}\}$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Any $f \in L^2(\mathbb{R}^n)$ can be written (uniquely) as $f = \sum_{e,j,k} \langle f, \Psi^{(e)}_{j,k}\rangle \Psi^{(e)}_{j,k}$

However, partial sums of this do not encode "low frequency" parts of f as Ψ is not smooth.

Eg 2: The Ψ, φ can be chosen to be Schwartz fns. and we consequently ~~by~~ as above we get a WAVELET basis of $L^2(\mathbb{R}^n)$.

Eg 3 (Characterizing Besov spaces): Let $\alpha_{p,f}(h) = \sup_{\ell \in B_0(h)} \frac{\|f(\cdot+\ell) - f(\cdot)\|_{L^p}}{h^\alpha}$

$f \in B^\alpha_{p,q} \iff \int_0^\infty (\alpha_{p,f}(h))^q \frac{dh}{h} < +\infty$

Let $\varphi_h(\cdot) = \frac{1}{h^n}\varphi\left(\frac{\cdot}{h}\right)$ be an approximation of the identity, $f_h = f * \varphi_h$

Now $\|f_h - f\|_{L^p} = \left\|\int_{B(x,h)} (f(y) - f(x))\varphi_h(x-y)\,dy\right\|_{L^p} \underset{(\text{young})}{\leq} \alpha_{p,f}(h)\, h^\alpha$

Similarly, $\|\partial^\beta f_h\|_{L^p} \leq \alpha_{p,f}(h)\, h^{-m+\alpha}$, where β is a multi-index with $|\beta| = m$.

Note for $h' < h$, $\alpha^p_{p,f}(h') \leq \alpha^p_{p,f}(h)\left[\frac{h}{h'}\right]^{\alpha p}$

Let $\tilde{\alpha}(h) = \alpha_{p,f}(2^j)$ for $2^{j-1} < h \leq 2^j$. So $2^{-\alpha}\tilde{\alpha}\left(\frac{h}{2}\right) \leq \alpha_{p,f}(h) \leq 2^\alpha \tilde{\alpha}(h)$

$\int_0^\infty (\alpha_{p,f}(h))^p \frac{dh}{h} = \sum_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} (\alpha_{p,f}(h))^q \frac{dh}{h} \approx \sum_{j \in \mathbb{Z}} \tilde{\alpha}(2^j)^q < +\infty \iff (\alpha_{p,f}(2^j))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$

$f \in B^\alpha_{p,q} \iff \gamma f = f_0 + \underbrace{(f_{1/2} - f_0)}_{g_1} + \underbrace{(f_{1/4} - f_0)}_{g_2} + \cdots$ s.t. $\|g_j\|_{L^p} \leq \varepsilon_j 2^{-j\alpha}$

$\exists f_0, g_1, g_2, \dots$ $\quad \|\partial^\beta g_j\|_{L^p} \leq \varepsilon_j 2^{-j(\alpha - m)}$, $|\beta| = m$

$(\varepsilon_j)_{j=1}^\infty \in \ell^q$

Put $g_j = \sum_{e,k} \langle f, \Psi^{(e)}_{j,k}\rangle \Psi^{(e)}_{j,k}$, so $\|g_j\|^p_{L^p} \sim \sum_{e,k} |\langle f, \Psi^{(e)}_{j,k}\rangle|^p\, 2^{nj(\frac{1}{2} - \frac{1}{p})p}$

$f \in B^\alpha_{p,q} \iff \left(\left(\sum_{e,k} |\langle f, \Psi^{(e)}_{j,k}\rangle|^p\right)^{1/p} 2^{nj(\frac{1}{2}-\frac{1}{p}) + j\alpha}\right)_{j=1}^\infty \in \ell^q$.