Harmonic functions and nodal sets

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Preface

We discuss the subject of nodal sets of solutions to partial differential equations (PDEs). The nodal set of a function is the set on which the function vanishes. If the PDE is elliptic such as the Laplace equation, then the nodal set of its non-trivial solution is consisted of hypersurfaces of co-dimension one (except at possibly a singular set of co-dimension greater than one). Our primary interest is the size of the nodal set in the hypersurface measure (i.e., co-dimension one Hausdorff measure).

One of the most influential problems on this subject is S. T. Yau's nodal size conjecture [**Y**]: Let $\mathbb{M}$ be an n-dim compact manifold and Δ be the Laplacian on $\mathbb{M}$. Then there are positive constants c_1 and c_2 depending only on $\mathbb{M}$ such that

$$c_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}(\phi)) \leq c_2\sqrt{\lambda}$$

for all Laplacian eigenfunctions ϕ, $-\Delta\phi = \lambda\phi$. Here, $\mathcal{N}(\phi) = \{x \in \mathbb{M} : \phi(x) = 0\}$ is the nodal set of ϕ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dim Hausdorff measure.

The methods in the study of nodal sets can be roughly divided to the complex method and the real method.

The complex method was developed by H. Donnelly, C. Fefferman, F.-H. Lin, Q. Han, S. Zelditch, etc, in the 1980s-1990s. That is, if the elliptic PDE has analytic coefficients, then the solutions are analytic and can be extended to holomorphic functions in a complex neighborhood of the domain. The complex analysis then provides powerful estimates of the nodal sets in the complex domain, which are then used to establish the bounds for the functions in the real domain. In particular, on a manifold with analytic metric, the Laplacian is an elliptic partial differential operator with analytic coefficients. Using the complex method, Yau's nodal size conjecture on analytic manifolds was proved [**DF1, Li**]. The nodal set theory of the complex method is relatively complete, see Han-Lin [**HL2**] and Zelditch [**Z**] for a systematic treatment. However, it does not apply to the smooth setting[i].

The real method was initiated by A. Logunov and E. Malinnikova [**Lo1, Lo2, LM1**] in 2016 and is still undergoing rapid development. It applies to elliptic PDEs with smooth coefficients. In particular, Logunov proved that on smooth manifolds,

$$c_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}(\phi)) \leq c_2\lambda^{\alpha},$$

in which $\alpha > 1/2$ is a constant that depends only on the dimension of the manifold. That is, the lower bound in Yau's nodal size conjecture is proved, while the upper bound leaves room for further improvement. (See Logunov-Malinnikova [**LM3**] for a complete history including earlier results.)

[i]The nodal size estimates in [**DF1, Li**] use the complex method and therefore do not apply to the smooth setting. However, we shall point out that certain results (such as the growth rate estimates of Laplacian eigenfunctions) in [**DF1, Li**] remain valid in the smooth setting. In addition, the (non-sharp) upper bound of nodal size of Laplacian eigenfunctions in [**DF2**] applies to smooth surfaces (i.e., $n = 2$ in Yau's nodal size conjecture).

In the framework of the real method, a Laplacian eigenfunction ϕ in $\mathbb{M}$ is lifted to a function u in $\mathbb{M}\times\mathbb{R}$ by $u(x,t)=\phi(x)e^{\sqrt{\lambda}t}$. We see that u solves an elliptic PDE with smooth coefficients:

$$(\Delta_x+\partial_t^2)u(x,t)=0 \quad \text{ in } \mathbb{M}\times\mathbb{R}.$$

Furthermore, the nodal set of u is a continuous copy of the one of ϕ: $\mathcal{N}(u)=\mathcal{N}(\phi)\times\mathbb{R}$. So their nodal sizes are related in a simple fashion. The lower and upper bounds of the nodal size of ϕ are reduced to the corresponding estimates of u, i.e., the nodal size estimates of solutions to elliptic PDEs.

This book grows from the notes I wrote for the Student Analysis Seminars at California State University Northridge (CSUN) since 2016. It was used in the research topic course, *Harmonic functions and nodal sets*, that I taught at CSUN in Fall 2019. The students in the seminars and the topic course are mostly undergraduate seniors and master's students. The book is therefore designed to assume as little prerequisite as possible, yet to reach as far as possible in the current research of nodal sets.

Based on these considerations, we focus on Logunov's upper bound λ^α, $\alpha>1/2$, of the nodal size of Laplacian eigenfunctions [**Lo1**] in this book[i]. It follows from the upper bound of the nodal size of solutions to elliptic PDEs with smooth coefficients. To simplify the presentation even further, we only discuss the Laplace equation in the Euclidean spaces as the prototype of elliptic PDEs. Of course, the Laplace equation has constant coefficients so the upper bound of the nodal size of their solutions (i.e., the harmonic functions) is already known by the complex method. However, our main objective is to explain the framework of the real method. Moderate modification of the one for the Laplace equation applies to elliptic PDEs. Moreover, any improvement of the real method of Logunov-Mallinikova for the Laplace equation will likely be applicable to general elliptic PDEs as well.

We organize the book as follows. In Chapters 1, we begin from the background of the nodal set subject and the language of multi-variable calculus. In Chapter 2, we prove the fundamental properties of the harmonic functions including the mean value theorems and maximal principle. In Chapter 3, we discuss the more advanced tools in the real method to study the nodal sets of harmonic functions, most notably, monotonicity and additivity of the frequency function and the doubling index. (Both of these two quantities characterize the growth rates of the harmonic function.) In Chapter 4, we apply these tools to prove the upper bounds of the nodal sizes of harmonic functions and Laplacian eigenfunctions.

I want to thank the participants of the Student Analysis Seminars at CSUN: Aida Britton, Kiarash Jannati, Michael Murray, Delfino Nolasco, Leah Schulman, and Chuong Tran. I also want to thank the students in the research topic course at CSUN in Fall 2019: Erik Arutyunyan, Armenuhi Barakezyan, Aida Britton, Masoud Eshaghinaseabadi, Paul Estrada, Shahdad Farahani, Kiarash Jannati, Mei Lim, Courtney van der Linden, Delfino Nolasco, Gialuka Raffel, and Chuong Tran.

[i]Logunov's sharp lower bound $\sqrt{\lambda}$ [**Lo2**] is not covered in this book. However, some important components of the proof in [**Lo2**], such as the monotonicity and additivity formulas in Chapter 3, overlap with the ones for the upper bound in this book.

Contents

CHAPTER 1

Introduction and preliminaries

1.1. Introduction

Denote the Laplacian in $\mathbb{R}^n$ as

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain.

Definition (Laplacian eigenfunctions). Let $\phi \in C^2(\Omega)$. We say that ϕ is a Laplacian eigenfunction if there exists $\lambda \in \mathbb{R}$ such that

$$-\Delta\phi(x) = \lambda\phi(x) \quad \text{for all } x \in \Omega, \tag{1.1}$$

in which we call λ the eigenvalue of ϕ. In addition, if $\phi(x) = 0$ for all $x \in \partial\Omega$ (the boundary of Ω), then we say that u is a Dirichlet eigenfunction.

In this note, we are interested in the analytic and geometric properties of eigenfunctions, particularly in the limit as $\lambda \to \infty$.

Example (Intervals). Let $\Omega = (0, L)$. Then the Dirichlet eigenfunctions are

$$\phi_k(x) = \sin\left(\frac{k\pi}{L}x\right) \qquad \text{with eigenvalue } \lambda_k = \left(\frac{k\pi}{L}\right)^2,\ k = 1, 2, 3, ...$$

That is, in the 1-dim domain, i.e., an interval, the Dirichlet eigenfunctions are explicitly represented by sine functions, of which we know all the analytic and geometric properties.

Example (Squares). Let $\Omega = (0, \pi) \times (0, \pi) \subset \mathbb{R}^2$. Then we can still build Dirichlet eigenfunctions from the sine functions:

$$\phi_{k,j}(x, y) = \sin(kx)\sin(jy) \quad \text{with eigenvalue } \lambda_{k,j} = |k|^2 + |j|^2,$$

in which $k, j \in \mathbb{Z}$, for example, $\phi_{4,7}(x, y) = \sin(4x)\sin(7y)$ with eigenvalue 65. These eigenfunctions are called the basic eigenmodes. However, unlike the interval case, they do not exhaust all the Dirichlet eigenfunctions. For example, $\phi_{7,4}(x, y) = \sin(7x)\sin(4y)$ is also a Dirichlet eigenfunction with the same eigenvalue 65. Since the eigenfunction equation (1.1) is linear,

$$a_1 \sin(4x)\sin(7y) + a_2 \sin(7x)\sin(4y)$$

is always a Dirichlet eigenfunction (with eigenvalue 65) for any $a_1, a_2 \in \mathbb{R}$. Basic arithmetic shows that

$$65 = 4^2 + 7^2 = 8^2 + 1^2,$$

that is, 65 can be represented by sums of two squares in two different ways. Correspondingly, the set of Dirichlet eigenfunctions with eigenvalue 65 (which is called its eigenspace) is spanned by the basic eigenmodes

$$\{\sin(4x)\sin(7y), \sin(7x)\sin(4y), \sin(x)\sin(8y), \sin(8x)\sin(y)\}.$$

The analytic and geometric properties of eigenfunctions, as linear combinations of basic eigenmodes, are much less obvious than a single basic eigenmode (as in the interval case). See below for more discussion.

In summary, the eigenvalues λ in Ω are integers that can be written as sums of two squares, while its eigenspace is spanned by

$$\left\{\sin(kx)\sin(jy) : k, j \in \mathbb{N}, k^2 + j^2 = \lambda\right\}.$$

The number of basic eigenmodes with eigenvalue λ (called the multiplicity of λ) is related to the number of representations of sums of two squares of λ, which is a classical problem in number theory with a long history, see for example, Grosswald [**Gr**].

Example (Rectangles). Let $\Omega = (0, L) \times (0, M) \subset \mathbb{R}^2$. Then the basic eigenmodes are

$$\phi_{k,j}(x, y) = \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{j\pi}{M}y\right) \quad \text{with eigenvalue } \lambda_{k,j} = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{j\pi}{M}\right)^2, \; k, j \in \mathbb{Z}.$$

The Dirichlet eigenfunctions can be written as the linear combinations of these basic eigenmodes with the same eigenvalue:

$$\sum_{\left(\frac{k\pi}{L}\right)^2 + \left(\frac{j\pi}{M}\right)^2 = \lambda} a_{k,j} \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{j\pi}{M}y\right), \quad \text{where } a_{k,j} \in \mathbb{R}.$$

One can similarly ask the analytic and geometric properties of these eigenfunctions. – There are a lot of unanswered questions already.

The main focus of research in this note is the study of nodal set:

Definition (Nodal sets). The nodal set $\mathcal{N}(\phi)$ of ϕ in Ω is the set of points where ϕ vanishes, i.e.,

$$\mathcal{N}(\phi) = \{x \in \Omega : \phi(x) = 0\}.$$

We are particularly interested in the size of the nodal set $\mathcal{N}(\phi)$, by which we call the nodal size of ϕ.

Example (Intervals). In $\Omega = (0, L)$, the Dirichlet eigenfunctions are

$$\phi_k(x) = \sin\left(\frac{k\pi}{L}x\right) \quad \text{with eigenvalue } \lambda_k = \left(\frac{k\pi}{L}\right)^2, \; k = 1, 2, 3, ...$$

The nodal set of ϕ_k is a collection of nodal points

$$\mathcal{N}(\phi_k) = \left\{\frac{L}{k} \cdot j : j = 1, ..., k-1\right\}.$$

Hence, the size of the nodal set of the k-th Dirichlet eigenfunction ϕ_k, i.e., the number of nodal points, is

$$\#\mathcal{N}(\phi_k) = k - 1 = \frac{L\sqrt{\lambda_k}}{\pi} - 1.$$

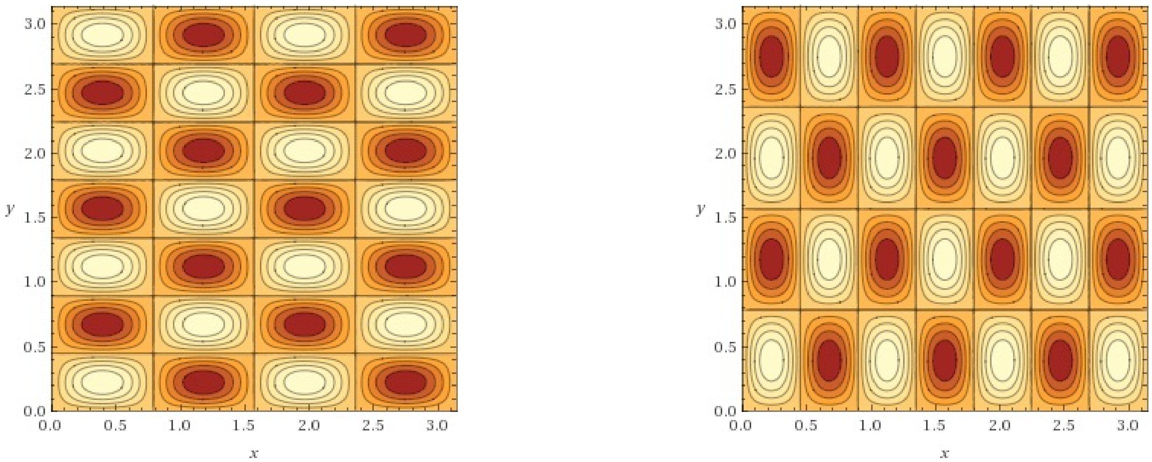
Therefore, the nodal size of ϕ_k is controlled by its eigenvalue λ_k explicitly. In fact, since we are interested in the limit as $\lambda_k \to \infty$,

$$\lim_{\lambda_k \to \infty} \frac{\#\mathcal{N}(\phi_k)}{\sqrt{\lambda_k}} = \frac{L}{\pi},$$

which is a quantity that depends only on the domain (actually, the size of the domain).

Example (Squares). In the square $\Omega = (0,\pi)\times(0,\pi)$, see the nodal portraits of $\sin(4x)\sin(7y)$ and $\sin(7x)\sin(4y)$, both of which have eigenvalue 65. Their nodal sets consist of vertical and horizontal lines. To measure the nodal size of these eigenmodes, we need to use the length:

$$\mathcal{L}(\mathcal{N}(\phi_{4,7})) = \mathcal{L}(\mathcal{N}(\phi_{7,4})) = 9\pi.$$



$\sin(4x)\sin(7y)$ $\qquad$ $\sin(7x)\sin(4y)$

Now consider the basic eigenmodes

$$\phi_{k,j}(x,y) = \sin(kx)\sin(jy).$$

The nodal set $\mathcal{N}(\phi_{k,j})$ is a collection of $|k|-1$ vertical lines and $|j|-1$ horizontal lines, i.e.,

$$\mathcal{N}(\phi_{k,j}) = \left\{\left\{\frac{1}{|k|}\right\}\times[0,\pi], ..., \left\{\frac{|k|-1}{|k|}\right\}\times[0,\pi]\right\} \cup \left\{[0,\pi]\times\left\{\frac{1}{|j|}\right\}, ..., [0,\pi]\times\left\{\frac{|j|-1}{|j|}\right\}\right\}.$$

Therefore,

$$\mathcal{L}(\mathcal{N}(\phi_{k,j})) = \pi(|k|+|j|-2).$$

Then

$$\frac{\mathcal{L}(\mathcal{N}(u_{k,j}))}{\sqrt{\lambda_{k,j}}} = \frac{\pi(|k|+|j|-2)}{\sqrt{|k|^2+|j|^2}}.$$

One sees from $\sqrt{p^2+q^2} \le p+q \le \sqrt{2}\sqrt{p^2+q^2}$ for $p,q\ge 0$ that

$$\liminf_{\lambda\to\infty} \frac{\mathcal{L}(\mathcal{N}(u_{k,j}))}{\sqrt{\lambda_{k,j}}} = \pi,$$

in which the limit is achieved by $u_{1,j}$ as $j\to\infty$ and by $u_{k,1}$ as $k\to\infty$, and

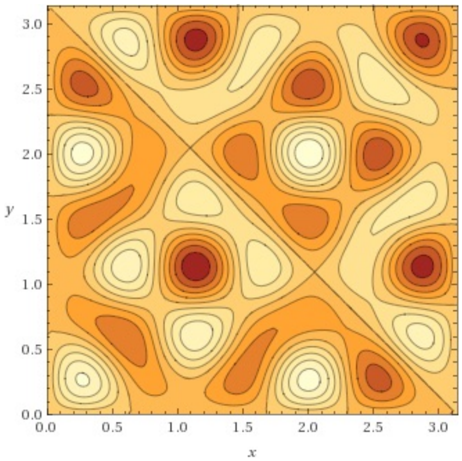
$$\limsup_{\lambda\to\infty} \frac{\mathcal{L}(\mathcal{N}(u_{k,j}))}{\sqrt{\lambda_{k,j}}} = \sqrt{2}\pi,$$

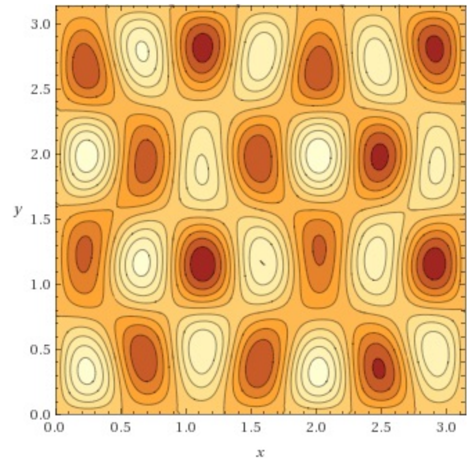
in which the limit is achieved by $u_{k,j}$ such that $k=j$ as $k,j\to\infty$.

It generalizes the 1-dim result to square, almost too perfectly! But recall that

$$a_1\sin(4x)\sin(7y) + a_2\sin(7x)\sin(4y)$$

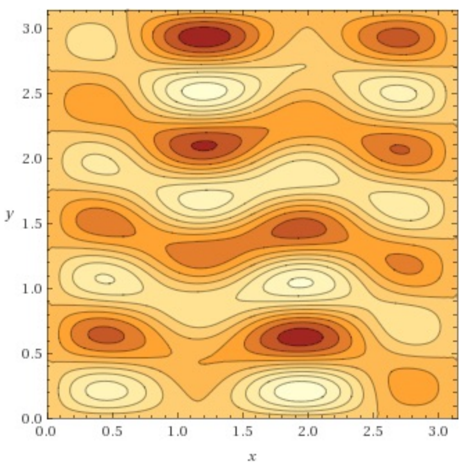
are Dirichlet eigenfunctions for all $a_1,a_2\in\mathbb{R}$, for which the eigenvalue is always 65. However, their nodal portrait are much more difficult to study:

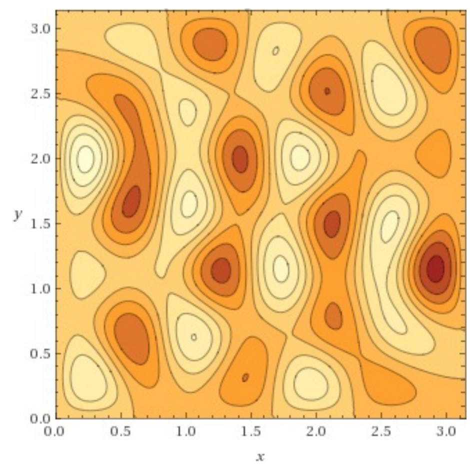




$\frac{1}{2}\sin(4x)\sin(7y) + \frac{1}{2}\sin(7x)\sin(4y)$ $\frac{1}{5}\sin(4x)\sin(7y) +$
$\frac{4}{5}\sin(7x)\sin(4y)$

We also have two more basic eigenmodes $\sin(x)\sin(8y)$ and $\sin(8x)\sin(y)$ with the same eigenvalue 65. Therefore, we need to consider linear combinations of more eigenmodes.





$\frac{1}{2}\sin(4x)\sin(7y)+\frac{1}{2}\sin(x)\sin(8y)$ $\frac{3}{10}\sin(4x)\sin(7y)+\frac{3}{10}\sin(7x)\sin(4y)+\frac{4}{10}\sin(8x)\sin(y)$

Measuring the total nodal length of these eigenfunctions then becomes a difficult question.

Question. How to measure the total nodal length of an eigenfunction ϕ in a domain Ω, in terms of the eigenvalues λ, particularly as $\lambda \to \infty$?

This question is challenging because

- The nodal portrait of eigenfunctions is complicated, i.e., the nodal curves are not explicitly computable, even in the case when the eigenfunctions are explicit (for example, the eigenfunctions are the linear combinations of sine functions in the rectangles).
- The eigenfunctions are not explicit in general domains (except special domains such as the rectangles shown above and the discs in Han-Murray-Tran [**HMT**]).
- As the eigenvalues $\lambda \to \infty$, the eigenfunctions oscillate more wildly, which results larger nodal sets and increasingly complicated nodal pattern.

The intuition from the interval case and basic eigenmodes in the square indicate that the nodal size of an eigenfunction is controlled by square root of the eigenvalue. Despite the different domains and eigenfunctions, S. T. Yau [**Y**] in the 1980s made the following far-reaching conjecture, vastly generalizing this intuition.[i]

CONJECTURE 1.1 (Yau's nodal size conjecture). *There exist* $c_1, c_2 > 0$ *depending only on* Ω *such that*

$$c_1\sqrt{\lambda} \leq \mathcal{L}(\mathcal{N}(\phi)) \leq c_2\sqrt{\lambda} \tag{1.2}$$

for all eigenfunctions ϕ *(with eigenvalue* λ*) in* Ω.

In fact, Yau's conjecture can be proposed in a much more general setting:

- the domain Ω is replaced by an n-dim Riemannian manifold $\mathbb{M}$ with $n \geq 2$, in which one can similarly define the Laplacian and its eigenfunctions;
- the nodal set $\mathcal{N}(\phi)$ of an eigenfunction ϕ in $\mathbb{M}$ is a union of $(n-1)$-dim hypersurfaces.[ii] The nodal size of ϕ is then measured by the total hypersurface measure of $\mathcal{N}(\phi)$, more rigorously defined by the $(n-1)$-dim Hausdorff measure $\mathcal{H}^{n-1}(\mathcal{N}(\phi))$.[iii]

In this setting, Yau's conjecture reads: There exist $c_1, c_2 > 0$ depending only on $\mathbb{M}$ such that

$$c_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}(\phi)) \leq c_2\sqrt{\lambda}$$

for all eigenfunctions ϕ (with eigenvalue λ) in $\mathbb{M}$.

Yau's nodal size conjecture has inspired several waves of research in the past 40 years, most notably Donnelly-Fefferman [**DF1, DF2**] in 1980s and Logunov [**Lo1, Lo2**] after 2016. Despite these important ideas being introduced to attack the conjecture, it is still only partially solved, in particular, the upper bound in (1.2) is open. See Logunov-Malinnikova [**LM3**] for the current state concerning the conjecture.

In this note, we focus on the most recent wave of research toward Yau's nodal size conjecture, Logunov [**Lo1, Lo2**] and Logunov-Malinnikova [**LM1**]. We will present (in fact, only a simplified version of the idea behind [**Lo1**])

$$\mathcal{H}^{n-1}(\mathcal{N}(\phi)) \leq c_2\lambda^{\alpha},$$

in which $c_2 = c_2(\Omega) > 0$ depends on the domain and $\alpha = \alpha(n) > 1/2$ depends only on n, the dimension of the domain. This is the best known upper bound so far, but since $\alpha > 1/2$, it leaves room for further improvement towards the one in Conjecture 1.1.

$\boxed{\textbf{Problems}}$.

1-1. Prove the inequalities

$$\sqrt{p^2+q^2} \leq p+q \leq \sqrt{2}\sqrt{p^2+q^2} \quad \text{for } p, q \geq 0.$$

Then provide the conditions that equality can be achieved.

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[i] Here is the original text of the conjecture in Yau [**Y**, Problem 74]:

"Let $\mathbb{M}$ be a compact surface. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the spectrum of $\mathbb{M}$ and $\{\phi_i\}$ be the corresponding eigenfunctions. For each i, the set $\{x|\phi_i(x) = 0\}$ is a one-dimensional rectifiable simplicial complex. Let L_i be the length of such a set. It is not difficult to prove that $\liminf_{i\to\infty} \sqrt{\lambda}^{-1}(L_i)$ has a positive lower bound depending only on the area of $\mathbb{M}$. It seems more difficult to find an upper bound of $\limsup_{i\to\infty} \sqrt{\lambda}^{-1}(L_i)$."

[ii] For example, if $\dim \mathbb{M} = 1$, i.e., in an interval, then $\mathcal{N}(\phi)$ is a collection of points; if $\dim \mathbb{M} = 2$, then $\mathcal{N}(\phi)$ is a union of curves; if $\dim \mathbb{M} = 3$, then $\mathcal{N}(\phi)$ is a union of surfaces.

[iii] For example, if $\dim \mathbb{M} = 1$, i.e., in an interval, then the nodal size of ϕ is the total number of nodal points, i.e., 0-dim Hausdorff measure; if $\dim \mathbb{M} = 2$, then the nodal size of ϕ is the total length of its nodal curves, i.e., 1-dim Hausdorff measure; if $\dim \mathbb{M} = 3$, then the nodal size of ϕ is the total surface measure of its nodal surfaces, i.e., 2-dim Hausdorff measure.

1.2. Preliminaries: multi-variable calculus

Recall the fundamental theorem of calculus: Let $f \in C^1([a,b])$. Then

$$\int_a^b f'(x)\,dx = f|_a^b = f(b) - f(a).$$

That is, the integral of f' in the interior of the domain $[a,b]$ equals the sum of f on the boundary (as two points a and b, with proper signs).

The generalization of the fundamental theorem of calculus to higher dimensions takes various forms, for example, Gauss-Green's theorem, divergence theorem, and Stokes' theorem. The additional complexity in higher dimensions, comparing with the 1-dim case, is natural and is due to

- a multivariable f has several partial derivatives, rather than one derivative,
- the boundary of a domain $\Omega \subset \mathbb{R}^n$ is an $(n-1)$-dim hypersurface, rather than two points.

In addition, recall the formula of integration by parts (IBP)

$$\int_a^b f'(x)g(x)\,dx = -\int_a^b f(x)g'(x)\,dx + fg|_a^b = -\int_a^b f(x)g'(x)\,dx + f(b)g(b) - f(a)g(a),$$

which we also generalize to higher dimensions.

Definition (Partial derivatives). Let $u : \mathbb{R}^n \to \mathbb{R}$ be differentiable.

- We use
$$\frac{\partial u}{\partial x_j} = \partial_{x_j} u = \partial_j u$$
to denote the partial derivative of u with respect to x_j, $j = 1, ..., n$.
- For any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_1, ..., \alpha_n \in \mathbb{N}$, we define the norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and
$$\partial^\alpha u = \partial_x^\alpha u = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\right) u.$$
- Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set. We define $C^k(\Omega)$ as the set of functions u such that $\partial_x^\alpha u \in C(\Omega)$ for all multiindices α with $|\alpha| \le k$ and $x \in \Omega$. In particular, $C^\infty(\Omega)$ is the set of infinitely differentiable (i.e., smooth) functions in Ω.

Definition (Normal vectors and normal derivatives). Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with (piecewise) smooth boundary. We say a vector ν is normal at $x \in \partial\Omega$ if it is orthogonal to the tangent plane of $\partial\Omega$ at x.

We usually choose the outward unit normal vector $\nu = (\nu_1, ..., \nu_n)$ in our computation, by which we simply refer as the (unit) normal.

Let $u \in C^1(\overline{\Omega})$. The (outward) normal derivative of u is defined as

$$\partial_\nu u = \frac{\partial u}{\partial \nu} = \nu \cdot \nabla u,$$

in which $\nabla u = (\partial_{x_1} u, ..., \partial_{x_n} u)$ is the gradient.

Example (Intervals). Let $\Omega = (a,b)$. Then the unit normal at the boundary point b is 1 and is -1 at the boundary point a. The normal derivative of $u \in C^1([a,b])$ is $u'(b)$ at b and is $-u'(a)$ at a.

Example (Rectangles). Let $\Omega = (0, L) \times (0, M) \subset \mathbb{R}^2$. Then the unit normal is

$$\begin{cases} (0,-1) & \text{on } (0,L) \times \{0\}, \\ (1,0) & \text{on } \{L\} \times (0,M), \\ (0,1) & \text{on } (0,L) \times \{M\}, \\ (-1,0) & \text{on } \{0\} \times (0,M). \end{cases}$$

The normal derivatives of $u(x, y) \in C^1(\overline{\Omega})$ are

$$\begin{cases} -\partial_y u & \text{on } (0,L) \times \{0\}, \\ \partial_x u & \text{on } \{L\} \times (0,M), \\ \partial_y u & \text{on } (0,L) \times \{M\}, \\ -\partial_x u & \text{on } \{0\} \times (0,M). \end{cases}$$

Example (Discs). Let $\Omega = \{(x, y) : x^2 + y^2 < R^2\}$. Then the unit normal at $(x, y) \in \partial\Omega$ is

$$\left(\frac{x}{R}, \frac{y}{R}\right).$$

The normal derivatives of $u(x, y) \in C^1(\overline{\Omega})$ is

$$\frac{x}{R}\partial_x u + \frac{y}{R}\partial_y u.$$

Remark. Observe that the unit normal ν depends on $x \in \partial\Omega$ and in fact defines a vector fields on $\partial\Omega$.

With the above necessary characterization of the boundary $\partial\Omega$, we state the following theorem which contains the fundamental theorem of calculus and integration by parts in their simplest forms.

THEOREM 1.2. *Let $u, w \in C^1(\overline{\Omega})$. Denote dS the hypersurface measure on $\partial\Omega$.*

(i). Fundamental theorem of calculus (in the j-th variable):

$$\int_\Omega \partial_{x_j} u \, dx = \int_{\partial\Omega} u\nu_j \, dS. \tag{1.3}$$

(ii). Integration by parts (in the j-th variable):

$$\int_\Omega w\partial_{x_j} u \, dx = -\int_\Omega u\partial_{x_j} w \, dx + \int_{\partial\Omega} uw\nu_j \, dS. \tag{1.4}$$

Remark. Let $\Omega = (a, b)$. Then by the above theorem and the fact that $\nu(a) = -1$ and $\nu(b) = 1$, we recover

(i). fundamental theorem of calculus:

$$\int_a^b u'(x) \, dx = u(a)\nu(a) + u(b)\nu(b) = u(b) - u(a),$$

(ii). integration by parts:

$$\begin{aligned} \int_a^b u'(x)w(x) \, dx &= \int_a^b u(x)w(x) \, dx + u(a)w(a)\nu(a) + u(b)w(b)\nu(b) \\ &= \int_a^b u(x)w(x) \, dx + u(b)w(b) - u(a)w(a). \end{aligned}$$

THEOREM 1.3 (Green's formulas). *Let $u, w \in C^2(\overline{\Omega})$. Then* *(i).*

$$\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \partial_\nu u\, dS,$$

(ii).

$$\int_\Omega \nabla u \cdot \nabla w\, dx = -\int_\Omega u\Delta w\, dx + \int_{\partial\Omega} u\partial_\nu w\, dS,$$

(iii).

$$\int_\Omega (u\Delta w - w\Delta u)\, dx = \int_{\partial\Omega} (u\partial_\nu w - w\partial_\nu u)\, dS.$$

PROOF.

(i). In (1.3), replacing u by $\partial_{x_j} u$, we have that

$$\int_\Omega \partial_{x_j}^2 u\, dx = \int_{\partial\Omega} \partial_{x_j} u\nu_j\, dS.$$

Hence,

$$\sum_{j=1}^n \int_\Omega \partial_{x_j}^2 u\, dx = \sum_{j=1}^n \int_{\partial\Omega} \partial_{x_j} u\nu_j\, dS.$$

The LHS,

$$\sum_{j=1}^n \int_\Omega \partial_{x_j}^2 u\, dx = \int_\Omega \sum_{j=1}^n \partial_{x_j}^2 u\, dx = \int_\Omega \Delta u\, dx,$$

while the RHS

$$\sum_{j=1}^n \int_{\partial\Omega} \partial_{x_j} u\nu_j\, dS = \int_{\partial\Omega} \sum_{j=1}^n \partial_{x_j} u\nu_j\, dS = \int_{\partial\Omega} \nabla u \cdot \nu\, dS = \int_{\partial\Omega} \partial_\nu u\, dS.$$

(ii). See Problem 1-4.

(iii). See Problem 1-4.

□

We also mention the following integration in polar coordinates that is frequently used in the note.

Definition (Balls and spheres). We denote $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ the (open) ball and $S(x, r) = \partial B(x, r) = \{y \in \mathbb{R}^n : |y - x| = r\}$ the sphere centered at x and with radius r. If there are spaces with different dimensions in question, then we denote $B^n(x, r) \subset \mathbb{R}^n$ and $S^{n-1}(x, r) \subset \mathbb{R}^n$ to distinguish them.

We also use $\mathbb{B}(x, r)$ to denote the closed ball $\overline{B(x, r)} = \{y \in \mathbb{R}^n : |y - x| \le r\}$.

THEOREM 1.4. *For any $u \in C(\mathbb{B}(x_0, R))$, we have that*

$$\int_{B(x_0,R)} u\, dx = \int_0^R \left(\int_{S(x_0,r)} u\, dS\right) dr.$$

Remark. Denote

$$f(t) = \int_{S(x_0,t)} u\, dS.$$

Then the above theorem is rewritten as

$$\int_{B(x_0,r)} u\, dx = \int_0^r f(t)\, dt \quad \text{for } 0 < r < R.$$

By the fundamental theorem of calculus (in 1-dim), we have that

$$\frac{d}{dr}\left(\int_0^r f(t)\,dt\right) = f(r).$$

Therefore, we have that

$$\frac{d}{dr}\left(\int_{B(x_0,r)} u\,dx\right) = f(r) = \int_{S(x_0,r)} u\,dS.$$

Definition (Volume of the unit ball). We denote α_n the volume of the unit ball $B(0,1) \subset \mathbb{R}^n$, i.e.,

$$\alpha_n = \int_{B(0,1)} dx,$$

for example, $\alpha_1 = 2$ and $\alpha_2 = \pi$.

By dilation, one sees that the volume of the ball $B(x,r) \subset \mathbb{R}^n$ is $\alpha_n r^n$.

Remark (Surface area of the unit sphere). The surface area of the unit sphere $\partial B(0,1) = n\alpha_n$. Indeed, let s be the surface area of the unit sphere in $\mathbb{R}^n$. Then the surface area of the sphere with radius r is sr^{n-1}. Therefore, by the above theorem

$$\alpha_n = \int_{B(0,1)} dx = \int_0^1 \int_{\partial B(0,r)} dS dr = \int_0^1 sr^{n-1}\,dr = sn,$$

which implies that $s = n\alpha_n$.

By dilation, one sees that the surface area of the sphere $S(x,r) \subset \mathbb{R}^n$ is $n\alpha_n r^{n-1}$.

$\boxed{\textbf{Problems}}$.

1-2. Find the unit normal at $(x_1, ..., x_n)$ on the boundary of a ball $B(0,R) = \{x \in \mathbb{R}^n : |x| < R\}$ and the normal derivatives of $u(x_1, ..., x_n)$.

1-3. Suppose that $u(x,y) = x^2 + y^2$ and $\Omega = B(0,r)$. Find

$$\int_\Omega \Delta u \quad \text{and} \quad \int_{\partial\Omega} \partial_\nu u,$$

and verify that they are equal, i.e., Green's formula in Theorem 1.3 (i).

1-4. Prove Theorem 1.3 (ii) and (iii). (Hint: For (ii), replace w by $\partial_{x_j} w$ in (1.4) and then sum with respect to j; For (iii), interchange u and w in (ii) and then subtract.)

1-5. Suppose that u and w are two Dirichlet eigenfunctions in Ω with different eigenvalues λ and μ. Prove that

$$\int_\Omega uv = 0.$$

(Hint: Use Green's formula in Theorem 1.3 (iii).)

1-6. Suppose that u is continuous at x.

(a). Prove that

$$\lim_{r\to 0} \frac{1}{\alpha_n r^n} \int_{B(x,r)} u(y)\,dy = u(x).$$

That is, the average of u in a ball $B(x,r)$ tends to $u(x)$ as $r \to 0$.

(b). Prove that

$$\lim_{r\to 0} \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,dy = u(x).$$

That is, the average of u on the sphere $\partial B(x,r)$ tends to $u(x)$ as $r \to 0$.

From now on, we denote the average in a ball and on a sphere as

$$\fint_{B(x,r)} u\,dy = \frac{1}{\alpha_n r^n}\int_{B(x,r)} u\,dy \quad \text{and} \quad \fint_{\partial B(x,r)} u\,dS = \frac{1}{n\alpha_n r^{n-1}}\int_{\partial B(x,r)} u\,dS.$$

CHAPTER 2

Harmonic functions

The Laplacian eigenfunctions in the previous chapter are closely related to the harmonic functions.

Definition (Harmonic functions). Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that $u \in C^2(\Omega)$ is harmonic in Ω if u solves the Laplace equation

$$\Delta u(x) = \frac{\partial^2 u(x)}{\partial x_1^2} + \cdots + \frac{\partial^2 u(x)}{\partial x_n^2} = 0 \quad \text{for all } x \in \Omega. \tag{2.1}$$

On one hand, harmonic functions can be viewed[i] as eigenfunctions with eigenvalue $\lambda = 0$ in (1.1). On the other hand, eigenfunctions in $\mathbb{R}^n$ can be "lifted" to harmonic functions in $\mathbb{R}^{n+1}$ by a simple procedure described as follows.

Suppose that $\phi \in C^2(\Omega)$ satisfies that $-\Delta_{\mathbb{R}^n}\phi(x) = \lambda\phi(x)$ for $\lambda > 0$. Write

$$u(x,t) = \phi(x)e^{\sqrt{\lambda}t}.$$

Then we readily check that

$$\begin{aligned}
\Delta_{\mathbb{R}^{n+1}} u(x,t) &= \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial t^2}\right)\phi(x)e^{\sqrt{\lambda}t} \\
&= e^{\sqrt{\lambda}t}\Delta_{\mathbb{R}^n}\phi(x) + \phi(x)\frac{\partial^2}{\partial t^2}e^{\sqrt{\lambda}t} \\
&= -e^{\sqrt{\lambda}t}\lambda\phi(x) + \phi(x)\lambda e^{\sqrt{\lambda}t} \\
&= 0.
\end{aligned}$$

Therefore, $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$ is harmonic in $\Omega \times \mathbb{R}$.

Moreover, the nodal sets of ϕ and of u are also related in a simple fashion. Indeed, since the exponential factor never vanishes,

$$\mathcal{N}(u) = \mathcal{N}(\phi) \times \mathbb{R}.$$

That is, $\mathcal{N}(u)$ is a continuous copy of $\mathcal{N}(\phi)$ into a higher-dimensional space. Hence, if $\mathcal{N}(\phi)$ is an $(n-1)$-dim hypersurface in Ω, then $\mathcal{N}(u)$ is an n-dim hypersurface in $\Omega \times \mathbb{R}$. Furthermore,

$$\mathcal{H}^n\left(\mathcal{N}(u) \cap (\Omega \times [a,b])\right) = (b-a) \cdot \mathcal{H}^{n-1}\left(\mathcal{N}(\phi) \cap \Omega\right),$$

i.e., the nodal size of u in $\Omega \times [a,b]$ equals $(b-a)$ times the nodal size of ϕ in Ω.

We thus shift our focus of the nodal size problem of eigenfunctions in Conjecture 1.1 to the one of harmonic functions. This chapter is devoted to the study of the basic analytic properties of harmonic functions. The main references are Evans [**E**, §2.2] and Han-Lin [**HL1**, Chapter 1].

The partial differential equation (PDE) (2.1) is a prototype of a large class of second order elliptic PDEs. Indeed, let $A = A(x) = [A(x)_{jk}]$ be an $n \times n$ matrix function, i.e., the entries

[i] This is not quite correctly though, since eigenvalues are usually required to be nonzero.

$A(x)_{jk}$ are real-valued functions of x. Then the PDE

$$\operatorname{div}(A\nabla u)(x) = \sum_{j=1}^{n} \partial_{x_j}\left(\sum_{k=1}^{n} A(x)_{jk}\partial_{x_k} u(x)\right) = 0 \tag{2.2}$$

is said to be elliptic (in the divergence form) if the matrix $A(x)$ is uniformly elliptic:

$$\Lambda^{-1}|\xi|^2 \leq \sum_{j,k=1}^{n} A(x)_{jk}\xi_j\xi_k \leq \Lambda|\xi|^2 \quad \text{with some } \Lambda > 0,$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^m$. We call $\operatorname{div}(A\nabla\cdot)$ a second order elliptic partial differential operator.

Example.

- The Laplace equation (2.1) is an elliptic PDE with A being the identity matrix.
- The Laplacian in a Riemannian manifold is a second order elliptic partial differential operator with A depending on the Riemannian metric.
- Most of the properties of harmonic functions, including the maximal principle and Harnack's inequality in this section, generalize to all second order elliptic partial differential operators. See again [**E, HL1**] for more details.

Problems.

2-1. Prove that $\Phi(x) = \log|x|$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

2-2. Prove that $\Phi(x) = |x|^{2-n}$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ for $n \geq 3$.

2.1. Mean value theorems

Suppose that u is harmonic. Then the important mean value theorems declare that $u(x)$ equals the average of u over the sphere $S(x,r)$, as well as the average of u over the ball $B(x,r)$. These implicit formulas involving u generates a remarkable number of consequences.

THEOREM 2.1 (Mean value theorem). *Let $u \in C^2(\Omega)$ be harmonic. Then*

$$u(x) = \fint_{S(x,r)} u\,dS = \fint_{B(x,r)} u\,dy \tag{2.3}$$

for each $B(x,r) \subset \Omega$.

PROOF.

(i). Set

$$\begin{aligned}
\phi(r) &= \fint_{S(x,r)} u\,dS \\
&= \frac{1}{n\alpha_n r^{n-1}}\int_{S(x,r)} u(y)\,dS(y) \\
&= \frac{1}{n\alpha_n r^{n-1}}\int_{S(0,r)} u(x+z)\,dS(z) \\
&= \frac{1}{n\alpha_n}\int_{S(0,1)} u(x+rw)\,dS(w).
\end{aligned}$$

Then using the Green's formulas in Theorem 1.3

$$\phi'(r) = \frac{1}{n\alpha_n}\int_{S(0,1)} \nabla u(x+rw)\cdot w\,dS(w)$$

$$\begin{aligned}
&= \frac{1}{n\alpha_n r^{n-1}} \int_{S(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y) \\
&= \frac{1}{n\alpha_n r^{n-1}} \int_{S(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy \\
&= 0.
\end{aligned}$$

Hence, $\phi(r)$ is constant and so

$$\phi(r) = \lim_{r\to 0} \phi(r) = \lim_{r\to 0} ⨍_{S(x,r)} u\, dS = u(x),$$

by Problem 1-6 since u is continuous.

(ii). Using the polar coordinates,

$$\begin{aligned}
⨍_{B(x,r)} u(y)\, dy &= \frac{1}{\alpha_n r^n} \int_{B(x,r)} u(y)\, dy \\
&= \frac{1}{\alpha_n r^n} \int_0^r \int_{S(x,s)} u(y)\, dS(y) ds \\
&= \frac{1}{\alpha_n r^n} \int_0^r n\alpha_n s^{n-1} u(x)\, ds \\
&= u(x).
\end{aligned}$$

□

The converse of the mean value theorem is also true. Here we present the first version.

THEOREM 2.2 (Converse to mean value property, Take I.). *If $u \in C^2(\Omega)$ satisfies*

$$u(x) = ⨍_{S(x,r)} u\, dS = ⨍_{B(x,r)} u\, dy$$

for each $B(x,r) \subset \Omega$, then u is harmonic.

PROOF. Suppose that $\Delta u(x) \neq 0$ for some $x \in \Omega$. If $\Delta u(x) > 0$, then there exists $B(x,r) \subset \Omega$ such that $\Delta u > 0$ in $B(x,r)$ since Δu is continuous. Denote ϕ as before. Then

$$0 = \phi'(r) = \frac{1}{n\alpha_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dS(y) > 0,$$

which is not possible.

The case when $\Delta u(x) < 0$ can be treated similarly. □

$\boxed{\textbf{Problems}}$.

2-1. We say u is subharmonic if

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

(a). Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex, i.e., $\phi'' \geq 0$. Prove that $w = \phi(u)$ is subharmonic if u is harmonic.

(b). Prove that $w = |\nabla u|^2$ is subharmonic if u is harmonic.

2-2. Suppose that $u \in C^2(\Omega)$ is subharmonic. Prove that

$$u(x) \leq \fint_{B(x,r)} u(y)\,dy \quad \text{for all } B(x,r) \subset \Omega.$$

2.2. Maximal principles and uniqueness

The maximal principle of a function u (usually as the solution to some PDE) indicates that the maximal (or minimal) value of u in a domain Ω must be attained on the boundary. It usually has two versions.

(i). [Weak version]:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii). [Strong version]: In addition, if the maximal value of u is attained in the interior, then u must be constant.

The maximal principle the the harmonic functions is a consequence of the mean value theorem proved in the previous section.

THEOREM 2.3 (Maximum principles). *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω.*

(i). [Weak maximum principle]

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii). [Strong maximum principle] Furthermore, if Ω is connected and there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u,$$

then u is constant within Ω.

PROOF.

(i). We use (ii) to prove (i). First assume that Ω is connected. Suppose that $\max_{\overline{\Omega}}$ is not attained in Ω. Then $\max_{\overline{\Omega}}$ must be attained on $\partial\Omega$, i.e.,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u,$$

so we are done. Now suppose that $\max_{\overline{\Omega}}$ is attained in Ω, i.e., there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$. Hence, u must be constant in Ω by (ii) so

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Now assume that $\Omega_1, ..., \Omega_m$ are connected components of Ω. Then

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega}_j} u$$

for some $j = 1, ..., m$, which subsequently continues as

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega}_j} u = \max_{\partial\Omega_j} u \leq \max_{\partial\Omega} u.$$

Therefore,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u,$$

since $\max_{\overline{\Omega}} u \geq \max_{\partial\Omega} u$ is obvious.

(ii). Denote

$$M = \max_{\overline{\Omega}} u \quad \text{and} \quad \Omega_M = \{x \in \Omega : u(x) = M\}.$$

We first show that Ω_M is open. Pick $x_0 \in \Omega_M$. Then for $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, $B(x_0, r) \subset \Omega$. By the mean value theorem in Theorem 2.1,

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)\, dy.$$

That is,

$$0 = \fint_{B(x_0,r)} u(y)\, dy - M = \fint_{B(x_0,r)} (u(y) - M)\, dy \le 0,$$

since $u(y) \le M$. The equality holds only if $u(y) = M$ for all $y \in B(x_0, r)$, i.e., $B(x_0, r) \subset \Omega_M$. So Ω_M is open.

Since u is continuous, $\{x \in \Omega : u(x) > M\}$ and $\{x \in \Omega : u(x) < M\}$ are both relatively open in Ω. Therefore,

$$\Omega_M = \Omega \setminus (\{x \in \Omega : u(x) > M\} \cup \{x \in \Omega : u(x) < M\})$$

is relatively closed in Ω. Hence, $\Omega_M = \Omega$ since Ω is connected, i.e., $u(x) = M$ for all $x \in \Omega$. □

Notice that $-u$ is also harmonic if u is harmonic. Moreover, $\max -u = -\min u$ so

(i).

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

(ii). Furthermore, if Ω is connected and there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{\overline{\Omega}} u,$$

then u is constant within Ω.

That is, the extreme (maximal and minimal) values of a harmonic function are always attained on the boundary. In convention, they are both referred as "maximal principle".

An immediate consequence of the maximal principle is the uniqueness of the solutions to the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Theorem 2.4 (Uniqueness). *Suppose that $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (2.4).*

Proof. Suppose that u_1 and u_2 both solve (2.4). Then $w = u_1 - u_2$ solves

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

That is, w is harmonic and equals 0 on the boundary. Therefore, by the maximal principles (for both maximal and minimal values), $w = 0$ in Ω, i.e., $u_1 = u_2$.

Alternatively, using the maximal principles twice, we have that $u_1 - u_2 \le 0$ and $u_2 - u_1 \le 0$. Therefore, $u_1 = u_2$. □

$\boxed{\textbf{Problems}}$.

2-3. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic. Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

2.3. Regularity and Liouville's theorem

LEMMA 2.5 (Mollifier). *Let $\eta(x) \in C_0^\infty(B(0,1))$ such that*

$$\eta(x) = \eta(|x|) \quad \text{and} \quad \int_{B(0,1)} \eta(x)\,dx = 1.$$

Define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Then the convolution

$$f_\varepsilon(x) := \eta_\varepsilon * f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)f(y)\,dy = \int_\Omega \eta_\varepsilon(x-y)f(y)\,dy$$

is smooth in $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ if f is continuous in Ω.

For example, one can set

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

in which the constant C is selected so that $\int \eta = 1$.

PROOF.

$$\partial_x^\alpha f_\varepsilon(x) = \int_{\mathbb{R}^n} \partial_x^\alpha \eta_\varepsilon(x-y)f(y)\,dy.$$

□

Remark. Notice that η_ε is supported in $B(0,\varepsilon)$ and $\int \eta_\varepsilon = 1$. Hence, $\eta_\varepsilon * f$ can be understood (roughly) as the average of f in $B(x,\varepsilon)$. In particular, if $\eta = \chi_{B(0,1)}$, the characteristic function of $B(0,1)$, then $\eta_\varepsilon = \chi_{B(0,\varepsilon)}/\varepsilon^n$ and $\eta_\varepsilon * f$ will be precisely the average of f in $B(x,\varepsilon)$ module a constant. But of course such choice of η is not a mollifier since it is not smooth. So one can understand the above definition of a mollifier as a "smooth" version of $\chi_{B(0,1)}$.

If f is harmonic, then we know from the mean value property that this average should be $f(x)$. We prove this in the following theorem.

THEOREM 2.6 (Smoothness). *If $u \in C(\Omega)$ satisfies the mean value property* (2.3) *for each ball $B(x,r) \subset \Omega$, then $u \in C^\infty(\Omega)$.*

PROOF. Set $x \in \Omega$. Then there is $\varepsilon > 0$ such that $x \in \Omega_\varepsilon$. We next show that $u(x) = u_\varepsilon(x)$ so u is smooth at x.

$$\begin{aligned} u_\varepsilon(x) &= \int_\Omega \eta_\varepsilon(x-y)u(y)\,dy \\ &= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dy \\ &= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\int_{S(x,r)} u(y)\,dS_y\right)dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{u(x)}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\int_{S(x,r)} 1\,dS_y\right)dr \\
&= u(x)\int_{B(x,\varepsilon)} \eta_\varepsilon(y)\,dy \\
&= u(x).
\end{aligned}
$$

Here, we used the mean value property that

$$\int_{S(x,r)} u(y)\,dS_y = u(x)\int_{S(x,r)} 1\,dS_y.$$

$\square$

THEOREM 2.7 (Converse to mean value property, Take II.). *If $u \in C(\Omega)$ satisfies*

$$u(x) = \fint_{S(x,r)} u\,dS = \fint_{B(x,r)} u\,dy$$

for each $B(x,r) \subset \Omega$, then u is harmonic.

PROOF. Because of Theorem 2.6, u is smooth. Then the theorem follows Theorem 2.2. $\square$

THEOREM 2.8 (Estimates on derivatives). *Assume that u is harmonic in Ω. Then*

$$|\partial^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}}\int_{B(x_0,r)} |u(y)|\,dy.$$

for each ball $B(x_0,r) \subset \Omega$ and each multiindex α of order $|\alpha| = k$. Here,

$$C_0 = \frac{1}{\alpha_n} \quad \text{and} \quad C_k = \frac{(2^{n+1}nk)^k}{\alpha_n} \quad \text{for } k \ge 1.$$

PROOF. We prove by induction on k.

(i). $k = 0$. By the mean value theorem 2.1, we have that

$$|u(x_0)| \le \frac{1}{\alpha_n r^n}\int_{B(x_0,r)} |u(y)|\,dy = \frac{C_0}{r^n}\int_{B(x_0,r)} |u(y)|\,dy.$$

(ii). $k = 1$. Notice that $\partial_{x_i} u \in C^\infty(\Omega)$ and is also harmonic. Applying the mean value theorem 2.1 to $\partial_{x_i} u$ on $B(x_0, r/2)$, we have that

$$\partial_{x_i} u(x_0) = \frac{2^n}{\alpha_n r^n}\int_{B(x_0,r/2)} \partial_{x_i} u(y)\,dy = \frac{2^n}{\alpha_n r^n}\int_{S(x_0,r/2)} u(y)\nu_i(y)\,dS_y.$$

Here, ν is the normal on $S(x_0, r/2)$ and we used the fundamental theorem of calculus in the i-th variable (1.3). Then

$$|\partial_{x_i} u(x_0)| \le \frac{2n}{r}\max_{S(x_0,r/2)} |u(y)|.$$

Now, if $y \in B(x_0, r/2)$, then $B(y, r/2) \subset B(x_0, r)$. Using Step (i),

$$|u(y)| \le \frac{2^n}{\alpha_n r^n}\int_{B(y,r/2)} |u(z)|\,dz \le \frac{2^n}{\alpha_n r^n}\int_{B(x_0,r)} |u(z)|\,dz.$$

Hence,

$$|\partial_{x_i} u(x_0)| \le \frac{2n}{r}\max_{S(x_0,r/2)} |u(y)| \le \frac{2^{n+1}n}{\alpha_n r^{n+1}}\int_{B(x_0,r)} |u(z)|\,dz.$$

(iii). Assume now $k \geq 2$ and the theorem holds for all balls in Ω and each multiindex of order $\leq k-1$. Fix $B(x_0, r) \subset \Omega$ and let α be a multiindex with $|\alpha| = k$. Then $\partial^\alpha u = \partial_{x_i}(\partial^\beta u)$ for some $i = 1, ..., n$ and $|\beta| = k-1$. Compute that

$$|\partial^\alpha u(x_0)| \leq \frac{kn}{r} \max_{S(x_0, r/k)} |\partial^\beta u(y)|.$$

Now, if $y \in B(x_0, r/k)$, then $B(y, (k-1)r/k) \subset B(x_0, r)$. By induction,

$$\begin{aligned} |\partial^\beta u(y)| &\leq \frac{C_k}{\left(\frac{(k-1)r}{k}\right)^{n+k-1}} \int_{B(y,(k-1)r/k)} |u(z)|\, dz \\ &\leq \frac{(2^{n+1}n(k-1))^{k-1}}{\alpha_n \left(\frac{(k-1)r}{k}\right)^{n+k-1}} \int_{B(x_0,r)} |u(z)|\, dz. \end{aligned}$$

Hence,

$$|\partial^\alpha u(x_0)| \leq \frac{kn}{r} \max_{S(x_0,r/k)} |u(y)| \leq \frac{(2^{n+1}nk)^k}{\alpha_n r^{n+k}} \int_{B(x_0,r)} |u(z)|\, dz.$$

□

THEOREM 2.9 (Liouville's theorem). *Suppose that u is harmonic and bounded in $\mathbb{R}^n$. Then u is constant.*

PROOF. Fix $x_0 \in \mathbb{R}^n$ and let $r > 0$. Apply the derivative estimate in Theorem 2.8. Then

$$|\partial_{x_i} u(x_0)| \leq \frac{2^{n+1}n}{\alpha_n r^{n+1}} \int_{B(x_0,r)} |u(z)|\, dz \leq \sup_{\mathbb{R}^n} |u| \cdot \frac{2^{n+1}n}{r} \to 0$$

as $r \to \infty$. Hence, $\partial_{x_i} u = 0$ for all $i = 1, ..., n$. So u is constant. □

2.4. Harnack's inequality

We next apply the mean value theorem to prove the Harnack's inequality. Let $V \Subset \Omega$ denote $\overline{V} \subset \Omega$ and is compact. Harnack's inequality then asserts that non-negative harmonic functions in Ω have comparable values in V if $V \Subset \Omega$ and is connected.

THEOREM 2.10 (Harnack's inequality). *For each connected open set $V \Subset \Omega$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \leq C \inf_V u$$

for all non-negative harmonic functions u in Ω.

One important aspect of the theorem is that the constant C depending only on V so is uniform for *all* harmonic functions. Thus,

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

for all points $x, y \in V$ and harmonic functions u. These inequalities assert that the values of a non-negative harmonic function within V are all comparable: u can not be very small or very large at any point of C unless u is very small or very large everywhere in V. The intuitive idea is that since V is a positive distance away from Ω, there is "room for the averaging effects of harmonic functions".

PROOF. Let $r = \operatorname{dist}(V, \partial\Omega)/4$. Choose $x, y \in V$ such that $|x - y| \leq r$. Then $B(y, r) \subset B(x, 2r) \subset \Omega$. By the mean value theorem

$$\begin{aligned} u(x) &= \frac{1}{\alpha_n (2r)^n} \int_{B(x,2r)} u(z)\,dz \\ &\geq \frac{1}{\alpha_n (2r)^n} \int_{B(y,r)} u(z)\,dz \\ &\geq \frac{1}{2^n} \cdot \frac{1}{\alpha_n r^n} \int_{B(y,r)} u(z)\,dz \\ &\geq \frac{1}{2^n} u(y). \end{aligned}$$

Since V is connected and $\overline{V}$ is compact, we can cover $\overline{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, ..., N$. Then

$$u(x) \geq \frac{1}{2^{n(N+1)}} u(y)$$

for all $x, y \in V$. □

A nice application of the Harnack's inequality is to prove that the nodal set of eigenfunctions with eigenvalue λ is "$\lambda^{-1/2}$ dense".

COROLLARY 2.11. *There is constant $c > 0$ such that for any Laplacian eigenfunction ϕ in $B(0,1)$ with eigenvalue $\lambda > 0$ and any $x \in B(0, 1/2)$, we have that* $\operatorname{dist}(x, \mathcal{N}(\phi)) \leq c\lambda^{-1/2}$.

PROOF. Suppose that there is some Laplacian eigenfunction ϕ in $B(0,1)$ with eigenvalue λ that does not change sign in some ball $B(x, r)$ for $x \in B(0, 1/2)$ and $r < 1/2$. Then without loss of generality, we assume that ϕ is positive. (If ϕ is negative, then consider $-\phi$.) Then the function

$$u(x, t) = \phi(x) e^{\sqrt{\lambda} t}$$

is harmonic and positve in $B(x, r) \times [-r, r]$. By the Harnack's inequality,

$$\sup_V u \leq C \inf_V u,$$

in which $V = B(x, r/2) \times [-r/2, r/2]$. Here, the constant C can be chosen to depend only on $B(0, 1/2)$. On the other hand, we have that

$$\sup_V u = e^{\sqrt{\lambda} r/2} \sup_{B(x,r/2)} \phi \quad \text{and} \quad \inf_V u = e^{-\sqrt{\lambda} r/2} \inf_{B(x,r/2)} \phi,$$

which implies that

$$e^{\sqrt{\lambda} r/2} \sup_{B(x,r/2)} \phi \leq C e^{-\sqrt{\lambda} r/2} \inf_{B(x,r/2)} \phi.$$

Hence,

$$e^{r\sqrt{\lambda}} \leq C,$$

therefore,

$$r \leq c\lambda^{-1/2} \quad \text{with } c = \log C.$$

□

2.5. Analyticity

Recall the Taylor's theorem in $\mathbb{R}^1$: Let $f \in C^m((a,b))$. Then for $x_0 \in (a,b)$,

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_m(x-x_0),$$

in which the reminder

$$R_m(x-x_0) = \frac{f^{(k)}(z)}{k!}(x-x_0)^m \quad \text{for some } z \text{ between } x_0 \text{ and } x.$$

Let $f \in C^\infty((a,b))$. If $R_m(x-x_0) \to 0$ as $m \to \infty$ in a neighborhood of x_0, then we say f is analytic at x_0 and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

We say that f is analytic in (a,b) if it is analytic at each point in (a,b).

Example (Analytic functions). The elementary functions such as polynomials, exponential and logarithmical functions, trigonometric functions, are analytic in their natural domains.

Example (Smooth but not analytic functions). The function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{|x|^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is smooth but not analytic at $x_0 = 0$. In particular, f is smooth so by the Taylor theorem

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_m(x-x_0),$$

in which the reminder R_m does not converge to 0 as $m \to \infty$. In fact, $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ so the power series is identically zero and $R_m(x-x_0) = f(x)$ for each $m \in \mathbb{N}$.

Remark. From the above discussion, if f is analytic at x_0, then $f^{(k)}(x_0) = 0$ for all $k \in \mathbb{N}$ (i.e., f vanishes of infinite order at x_0) implies that $f = 0$.

To generalize Taylor's theorem to higher dimensions, we need addition notations involving multiindices. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multiindex, in which $\alpha_1, ..., \alpha_n \in \mathbb{N}$. Recall that we define the norm of α as $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also define $\alpha! = \alpha_1! \cdots \alpha_n!$ and for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ that

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

THEOREM 2.12 (Taylor's theorem and analytic functions). *Let $f \in C^m(\Omega)$. Then for $x_0 \in \Omega$,*

$$f(x) = \sum_{k=0}^{m-1} \sum_{|\alpha|=k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + R_m(x-x_0),$$

in which the reminder

$$R_m(x-x_0) = \sum_{|\alpha|=m} \frac{\partial^\alpha f(x_0)(z)}{\alpha!}(x-x_0)^\alpha \quad \textit{in which } z = x_0 + t(x-x_0) \textit{ for some } t \in [0,1].$$

Let $f \in C^\infty(\Omega)$. If $R_m(x-x_0) \to 0$ as $m \to \infty$ in a neighborhood of x_0, then we say f is analytic at x_0 and write

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha.$$

We say that f is analytic in Ω if it is analytic at each point in Ω. The class of analytic functions in Ω is denoted as $C^\omega(\Omega)$.

Similarly as in $\mathbb{R}^1$, the elementary functions are analytic. Examples of smooth but not analytic functions include the mollifier in Section 2.3. In particular, if an analytic function vanishes at infinite order at x_0 must be identically zero.

THEOREM 2.13. *If u is harmonic in Ω, then u is analytic in Ω.*

We first need the following lemma.

LEMMA 2.14. *If u is harmonic in $B(x_0,R)$, then for any multiindex $|\alpha| = m$,*

$$|\partial^\alpha u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \cdot \max_{B(x_0,R)} |u|.$$

PROOF. Assume that it holds for all multiindices α such that $|\alpha| = 0, ..., m$. Consider $|\alpha| = m+1$. Then $\partial^\alpha = \partial_{x_i}\partial^\beta$ for some $i = 1, ..., n$ and multiindex $|\beta| = m$. Denote $r = (1-t)R \in (0,R)$ for $t \in (0,1)$ (to be chosen later). Applying the mean value theorem 2.1 to $\partial_{x_i}\partial^\beta u$ on $B(x_0,r)$, we have that

$$\partial^\alpha u(x_0) = \frac{1}{\alpha_n r^n}\int_{B(x_0,r)} \partial_{x_i}\partial^\beta u(y)\,dy = \frac{1}{\alpha_n r^n}\int_{S(x_0,r)} \partial^\beta u(y)\nu_i(y)\,dS_y.$$

Then

$$|\partial^\alpha u(x_0)| \le \frac{n}{r}\max_{S(x_0,r)} |\partial^\beta u| \le \frac{n}{r}\max_{B(x_0,r)} |\partial^\beta u|.$$

For each $y \in B(x_0,r)$, $B(y,R-r) \subset B(x_0,R)$. By induction,

$$|\partial^\beta u(y)| \le \frac{n^m e^{m-1} m!}{(R-r)^m} \cdot \max_{B(y,R-r)} |u| \le \frac{n^m e^{m-1} m!}{(R-r)^m} \cdot \max_{B(x_0,R)} |u|.$$

Hence,

$$\begin{aligned}
|\partial^\alpha u(x_0)| &\le \frac{n}{r}\max_{B(x_0,R)} |\partial^\beta u| \\
&\le \frac{n}{r} \cdot \frac{n^m e^{m-1} m!}{(R-r)^m} \cdot \max_{B(x_0,R)} |u| \\
&\le \frac{n^{m+1} e^{m-1} m!}{R^{m+1} t^m (1-t)} \cdot \max_{B(x_0,R)} |u| \\
&\le \frac{n^{m+1} e^{m} (m+1)!}{R^{m+1}} \cdot \max_{B(x_0,R)} |u|,
\end{aligned}$$

in which we take $t = m/(m+1)$ and observe that

$$\frac{1}{t^m(1-t)} = \left(1+\frac{1}{m}\right)^m (m+1) < e(m+1).$$

□

Proof of Theorem 2.13. Let $x_0 \in \Omega$. It suffices to prove that the reminder term $R_m(x-x_0) \to 0$ as $m \to \infty$ for $x \in B(x_0, r)$ with some $r > 0$. Fix $R = \operatorname{dist}(x_0, \partial\Omega)/2$. Then $B(x_0, 2R) \in \Omega$. For each $z \in B(x_0, R)$, $B(z, R) \subset B(x_0, 2R)$. By the previous lemma, if $|\alpha| = m$, then

$$|\partial^\alpha u(z)| \le \frac{n^m e^{m-1} m!}{R^m} \cdot \max_{B(z,R)} |u| \le \frac{n^m e^{m-1} m!}{R^m} \cdot \max_{B(x_0,2R)} |u|.$$

If $|x - x_0| < r$, then $|(x - x_0)^\alpha| \le r^m$. Furthermore, $\alpha! \le m!$. Therefore,

$$\begin{aligned} |R_m(x-x_0)| &= \left| \sum_{|\alpha|=m} \frac{\partial^\alpha f(x_0)(z)}{\alpha!} (x-x_0)^\alpha \right| \\ &\le \frac{n^m e^{m-1} r^m}{R^m} \cdot \max_{B(x_0,2R)} |u| \cdot \sum_{|\alpha|=m} \frac{m!}{\alpha!} \\ &\le \frac{n^m e^{m-1} r^m}{R^m} \cdot \max_{B(x_0,2R)} |u| \cdot n^m \\ &\le \left(\frac{n^2 e r}{R} \right)^m \cdot \max_{B(x_0,2R)} |u| \\ &\to 0, \end{aligned}$$

as $m \to \infty$ if we choose

$$r = \frac{R}{2n^2 e}.$$

Here, we used the following multinomial theorem. □

Remark (Multinomial theorem). We have that

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha,$$

which implies that (by taking $x_1 = \cdots = x_n = 1$)

$$n^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!}.$$

The multinomial coefficients $m!/\alpha!$ have a direct combinatorial interpretation, as the number of ways of depositing m distinct objects into n distinct bins, with α_1 objects in the first bin labelled for x_1, α_2 objects in the second bin labelled for x_2, and so on. To find such coefficient,

- Choose α_1 of the total m to be labeled x_1. This can be done $\binom{m}{\alpha_1}$ ways.
- From the remaining $m - \alpha_1$ items, choose α_2 to label x_2. This can be done $\binom{m-\alpha_1}{\alpha_2}$ ways.
- $\cdots$.
- From the remaining $m - \alpha_1 - \cdots - \alpha_{n-1}$ items, choose α_n to label x_n. This can be done $\binom{m-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n}$ ways.

Therefore, the coefficient equals

$$\begin{aligned} & \binom{m}{\alpha_1} \binom{m-\alpha_1}{\alpha_2} \cdots \binom{m-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n} \\ = \;& \frac{m!}{(m-\alpha_1)!\alpha_1!} \cdot \frac{(m-\alpha_1)!}{(m-\alpha_1-\alpha_2)!\alpha_2!} \cdot \cdots \cdot \frac{(m-\alpha_1-\cdots-\alpha_{n-1})!}{(m-\alpha_1-\cdots-\alpha_n)!\alpha_n!} \end{aligned}$$

$$
\begin{aligned}
&= \frac{m!}{\alpha_1! \cdots \alpha_n!} \\
&= \frac{m!}{\alpha!}.
\end{aligned}
$$

Definition (Vanishing orders). Let $f \in C^\infty(\Omega)$ and $f(x) = 0$. The vanishing order of f at x, denoted by $V_f(x)$, is defined as the largest integer d such that

$$\partial^\alpha f(x) = 0 \quad \text{for all } |\alpha| \le d,$$

We write $V_f(x) = \infty$ if $\partial_\alpha f(x) = 0$ for all multiindex α.

Example.

(1). Let $f(x) = (x-a)^k$ in $\mathbb{R}$. Then $V_f(a) = k - 1$.

(2). If f is a polynomial of one variable with degree k, the $V_f(x) \le k-1$ for all $x \in \mathbb{R}$.

(3). The fundamental theorem of algebra states that a polynomial of degree k has at most k zeros. This can be restated in the language of vanishing orders:

$$\sum_{x \in \mathbb{R}: f(x)=0} (V_f(x) + 1) \le k.$$

(4). If f is a polynomial of n variables and degree k, then $V_f(x) \le k-1$ for all $x \in \mathbb{R}^n$. However, there is no corresponding version of the fundamental theorem of algebra as above.

(5). The function

$$f(x) = \begin{cases} e^{-\frac{1}{|x|^2}} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has vanishing order ∞ at 0.

COROLLARY 2.15. *Non-zero analytic functions have finite vanishing orders. That is, if f is analytic and $f \ne 0$ in Ω, then $V_f(x) < \infty$ for all $x \in \Omega$.*

PROOF. Suppose that $V_f(x) = \infty$ for $x \in \Omega$. Then $\partial^\alpha f(x) = 0$ for all α and the Taylor expansion of f at x is identically zero. But f is analytic so is identically zero. $\square$

Remark.

- Corollary 2.15 is obviously false if f is only smooth, as shown in Example (5) above.
- If $V_f(x) = d$, then the Taylor series of f at x begins from the $(d+1)$-th term since the terms with lower orders are zero.

2.6. Cauchy uniqueness theorem

We know from the uniqueness theorem in Section 2.2 that a harmonic function u is uniquely determined in a bounded domain Ω if the Dirichlet data u is given on the boundary $\partial\Omega$. In particular, it suffices to show that $u = 0$ if u is harmonic and vanishes on the boundary $\partial\Omega$.

Cauchy uniqueness theory (in part) deals with the problem of whether a function (usually a solution to a PDE) is uniquely determined if the boundary data is specified only on a piece of the boundary rather than on the whole boundary, in particular, whether $u = 0$ in Ω if the boundary data is zero on a piece of the boundary.

First, the answer to this problem is negative if we only specify the Dirichlet data on a piece of the boundary. Consider the harmonic function $u = xy$, which vanishes on $\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$. However, $u \ne 0$ in any domain that contains a piece of $\{x = 0 \text{ or } y = 0\}$, for example, $\Omega = (0, 1) \times (0, 1)$.

Therefore, one needs more boundary data for the answer to be positive. We call $\partial_\nu u$ the Neumann data and $(u, \partial_\nu u)$ the Cauchy data (i.e., the Dirichlet and Neumann data) on the boundary. Then the Cauchy uniqueness theory for the harmonic functions in $\mathbb{R}^n$ states that $u = 0$ in a neighborhood of a hyperplane if the Cauchy data vanishes on the hyperplane. In this section, we mention a quantitative version of the Cauchy uniqueness theory.

THEOREM 2.16 (Quantitative Cauchy uniqueness theorem). *Let $Q = [-R/2, R/2]^n \subset \mathbb{R}^n$ be a cube. Denote F a face of Q. There exist constant $C > 0$ and $\alpha \in (0,1)$ such that if a harmonic function u in Q satisfies*

$$\begin{cases} |u(x)| \le 1, & \text{if } x \in Q, \\ |u(x)| \le \varepsilon, & \text{if } x \in F, \\ |\partial_\nu u(x)| \le \frac{\varepsilon}{R}, & \text{if } x \in F, \end{cases}$$

for some $\varepsilon \ge 0$, then

$$\sup_{\frac{1}{2}Q} |u| \le C\varepsilon^\alpha.$$

Here, $\frac{1}{2}Q = [-R/4, R/4]^n$ is the middle cube of Q with half side-length.

The proof can be found in Lin [**Li**, Lemma 4.3]. An immediate consequence is the Cauchy uniqueness theorem.

COROLLARY 2.17 (Cauchy uniqueness theorem). *Let $Q \subset \mathbb{R}^n$ be a cube. Denote F a face of Q. Suppose that $f \in C(Q)$ and $g, h \in C(\partial\Omega)$. Then there exists at most one solution $u \in C^2(Q)$ to the Cauchy problem*

$$\begin{cases} \Delta u = 0 & \text{in } Q, \\ u = f \text{ and } \partial_\nu u = h & \text{on } F. \end{cases}$$

PROOF. Suppose that u_1 and u_2 both solve the Cauchy problem in $C^2(Q)$. Let

$$M = \max_Q |u_1| + \max_Q |u_2|.$$

Then $M = 0$ implies that $u_1 = u_2$ in Q. Otherwise, $w = (u_1 - u_2)/M$ solves

$$\begin{cases} \Delta w = 0 & \text{in } Q, \\ w = \partial_\nu w = 0 & \text{on } F. \end{cases}$$

Moreover, $|w| \le 1$ in Q. By Theorem 2.16, $w = 0$ in $\frac{1}{2}Q$. But w is analytic by Theorem 2.13. Therefore, $w = 0$ in Q, i.e., $u_1 = u_2$. □

The other consequence of Theorem 2.16 is about the vanishing order of harmonic functions. Recall that if f has vanishing order d at x, then $\partial^\alpha f(x) = 0$ for all $|\alpha| \le d$. Harmonic functions can have arbitrarily high vanishing orders at a point, for example, $(x + iy)^k$ has vanishing order $k - 1$ at the origin for $k \in \mathbb{N}$.

We can similarly define the vanishing order of f on a set E as the largest number d such that $\partial^\alpha f(x) = 0$ for all $|\alpha| \le d$ and $x \in E$. Next we show that harmonic functions can not vanish on hyperplanes with arbitrarily high order. In fact, Cauchy uniqueness theorem asserts that the vanishing order of any harmonic function on hyperplanes is at most zero, unless it is identically zero. Indeed, if the vanishing order of a harmonic function u on a hyperplane is at least one, then $u = 0$ and $\partial_\nu u = 0$ on the hyperplane. Therefore, $u = 0$ by the Cauchy uniqueness theorem.

COROLLARY 2.18. *Let u be harmonic in Ω. Suppose that F is a piece of hyperplane F in Ω. Then the vanishing order of u on F is at most zero unless $u = 0$ in Ω.*

Remark. If u is not harmonic, then the above corollary is invalid. That is, there are functions that vanish to high orders on hyperplanes. For example, let $k \in \mathbb{N}$ and $u = x_1^k$. Then u has vanishing order $k-1$ on the hyperplane $\{x_1 = 0\}$.

CHAPTER 3

Doubling index and frequency function

Yau's conjecture 1.1 states that the nodal size of eigenfunction ϕ, $-\Delta\phi = \lambda\phi$, is comparable to $\sqrt{\lambda}$. For the harmonic functions, the appropriate quantity to control the nodal size is the "doubling index". Recall that $\mathbb{B}(x,r) = \overline{B(x,r)}$.

Definition (Doubling index). Let $u \in C^0(\Omega)$. For $\mathbb{B}(x,2r) \subset \Omega$, define the doubling index of u in $B(x,r)$ as

$$N_u(x,r) = \log_2 \left(\frac{\max_{\mathbb{B}(x,2r)} |u|}{\max_{\mathbb{B}(x,r)} |u|} \right).$$

If there is only one function in question, then we omit the subscript u and write $N(x,r)$; if the center x is also fixed in the discussion, then we simply write $N(r)$.

In this chapter, we study the properties of the doubling index of harmonic functions, with the focus on the following two questions:
(1). Monotonicity: For a fixed center x, is $N_u(x,r)$ monotone in r?
(2). Additivity: Let $B(x_j,r_j) \subset B(x,R)$ be disjoint. Is $N_u(x,R) \geq \sum_j N_u(x_j,r_j)$?
Our ultimate goal is to apply these properties to the nodal size estimates of harmonic functions.

Conjecture 3.1 (Upper bound of the nodal size of harmonic functions). *There exist positive constants $C = C(n)$ and $K = K(n)$ such that*

$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x,r)) \leq C \cdot r^{n-1} N_u(x, Kr)$$

for all harmonic functions u in $\mathbb{B}(x, 2Kr)$.

Here, we mention the upper bound only. See Section 3.2 for the conjecture of the lower bound and its solution in $\mathbb{R}^2$. These nodal size estimates for harmonic functions can also be asked for general elliptic PDEs (2.2): Let u be a solution to an elliptic PDE

$$\operatorname{div}(A\nabla u)(x) = \sum_{j=1}^n \partial_{x_j} \left(\sum_{k=1}^n A(x)_{jk} \partial_{x_k} u(x) \right) = 0.$$

Then do we have that

$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x,r)) \leq C \cdot r^{n-1} N_u(x, Kr)?$$

Here is a list of known results:

- Donnelly-Fefferman [**DF1**] proved the upper bound $Cr^{n-1}N_u$ for analytic elliptic PDEs, that is, the coefficients $A(x)$ are analytic in x. In particular, Conjecture 3.1 is proved since the Laplace equation $\Delta u = 0$ has constant coefficients. The proof of this upper bound in [**DF1**] uses the complex method, which does not apply to the smooth setting.
- Hardt-Simon [**HS**] proved the upper bound $Cr^{n-1}e^{N_u}$ for smooth elliptic PDEs.

- Logunov [**Lo1**] proved the upper bound $Cr^{n-1}N_u^\alpha$ with some constant $\alpha > 1$ for smooth elliptic PDEs. In this chapter, we prepare the necessary tools for the proof of this upper bound, through the example of the Laplace equation (despite the fact that better upper bound is known).

Moreover, the harmonic functions and eigenfunctions are closely related, in particular, $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$ is harmonic if ϕ is an eigenfunction with eigenvalue λ, see Chapter 2. Hence, the nodal size estimates of u naturally yields the information on the nodal size of ϕ as in Yau's nodal size conjecture 1.1. To this end, one needs to further connect the doubling index of u with the eigenvalue λ of ϕ. We discuss this relation in the next chapter.

3.1. Homogeneous harmonic polynomials

The nodal sets of polynomials are most regular among all functions. In particular, the fundamental theorem of algebra asserts that the number of nodal points of a single-variable polynomial is bounded above by its degree. The situation for the nodal sets of multi-variable polynomials is more complicated and we study some of their properties. Let $k \in \mathbb{N}$. We denote

- $\mathbb{P}_k$: the space of polynomials of degree k in $\mathbb{R}^n$,
- $\mathbb{G}_k$: the space of homogeneous polynomials of degree k in $\mathbb{R}^n$,
- $\mathbb{H}_k$: the space of homogeneous harmonic polynomials of degree k in $\mathbb{R}^n$.

Of course, these spaces are all linear vector spaces and $\mathbb{P}_k \supset \mathbb{G}_k \supset \mathbb{H}_k$. It is straightforward to deduce that

$$\dim \mathbb{P}_k = O_n\left(k^n\right), \quad \dim \mathbb{G}_k = O_n\left(k^{n-1}\right), \quad \text{and } \dim \mathbb{H}_k = O_n\left(k^{n-2}\right),$$

as the degree $k \to \infty$ and dimension n is fixed. See the following for more details.

In this section, we get some basic understanding of the doubling index, the degree, and the nodal size, via the study of homogeneous harmonic polynomials.

First, the space of polynomials of degree k in $\mathbb{R}^n$

$$\mathbb{P}_k = \operatorname{span}\{x^\alpha : |\alpha| \le k\},$$

in which $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex with norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Therefore,

$$\dim \mathbb{P}_k = \binom{n+k}{n}.$$

In particular, $\dim \mathbb{P}_k = O_n\left(k^n\right)$ as the degree $k \to \infty$ and dimension n is fixed.

Next, the space of homogeneous polynomials

$$\mathbb{G}_k = \operatorname{span}\{x^\alpha : |\alpha| = k\},$$

and therefore has dimension

$$\dim \mathbb{P}_k = \binom{n+k-1}{n-1} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}.$$

In particular, $\dim \mathbb{P}_k = O_n\left(k^{n-1}\right)$ as the degree $k \to \infty$ and dimension n is fixed.

A homogeneous polynomial of degree k can also be written in the spherical coordinates:

$$P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha = |x|^k \sum_{|\alpha|=k} c_\alpha \left(\frac{x}{|x|}\right)^\alpha = r^k \phi(z),$$

in which $r = |x| \in [0, \infty)$, $z = x/|x| \in S(0,1)$, and ϕ is a function on $S(0,1)$. The doubling index of homogeneous harmonic polynomials is obvious in $B(0,r)$:

$$N_P(0,r) = \log_2 \left(\frac{\max_{\mathbb{B}(0,2r)} |P|}{\max_{\mathbb{B}(0,r)} |P|} \right) = \log_2 \left(\frac{(2r)^k \cdot \max_{S(0,1)} |\phi|}{r^k \cdot \max_{S(0,1)} |\phi|} \right) = k,$$

in which k is the degree of P.

Lastly, for a homogeneous polynomial P of degree k in $\mathbb{R}^n$, ΔP is a homogeneous polynomial of degree $k-2$. This in turn determines that for $k \geq 2$,

$$\dim \mathbb{H}_k = \dim \mathbb{G}_k - \dim \mathbb{G}_{k-2} = \binom{n+k-1}{k} - \binom{n+k-3}{k-1}.$$

Moreover, $\dim \mathbb{H}_0 = 1$ (i.e., $\mathbb{H}_0$ consists of constant functions) and $\dim \mathbb{H}_1 = n$ (i.e., $\mathbb{H}_1$ is spanned by the degree one polynomials $\{x_1, ..., x_n\}$.) As $k \to \infty$, $\dim \mathbb{H}_k = O_n\left(k^{n-2}\right)$.

Example ($\mathbb{H}_k$ in $\mathbb{R}^2$). In $\mathbb{R}^2$,

$$\dim \mathbb{H}_k = 2.$$

Write $z = x + iy$ for $(x,y) \in \mathbb{R}^2$. Then $z^k = (x+iy)^k$ is harmonic and of homogeneous degree k. We then have that $\mathbb{H}_k$ is spanned by $\{\Re(z^k), \Im(z^k)\}$. For example,

$$\mathbb{H}_0 = \text{span}\,\{1\},$$
$$\mathbb{H}_1 = \text{span}\,\{x, y\},$$
$$\mathbb{H}_2 = \text{span}\,\{x^2 - y^2, xy\},$$
$$\mathbb{H}_3 = \text{span}\,\left\{x^3 - 3xy^2, 3x^2y - y^3\right\},$$

and so on. In polar coordinates though, they have much easier representation:

$$\mathbb{H}_k = \text{span}\,\left\{r^k \sin(k\theta), r^k \cos(k\theta)\right\} \quad \text{for } k \in \mathbb{N}.$$

Their nodal sets in $B^2(0,r)$ are easy to picture. Indeed, the nodal set of $r^k \sin(k\theta)$ or $r^k \cos(k\theta)$ is a union of k diameters that intersect at the origin with equal angles of π/k. Hence, $\mathcal{H}^1(\mathcal{N}(P_k) \cap B^2(0,r)) = 2rk$. (Recall that $B^n(x,r)$ denotes the n-dim ball.)

Example ($\mathbb{H}_k$ in $\mathbb{R}^3$). In $\mathbb{R}^3$,

$$\dim \mathbb{H}_k = 2k+1.$$

One can find a basis of $\mathbb{H}_k$ using Legendre functions (see for example, Han-Lin [**HL2**, §2.1].) There are rather complicated, furthermore, the nodal sets are not straightforward to picture. For example, using the spherical coordinates (r, θ, φ) for $r \geq 0$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$, it is known that

$$r^k f(\cos\theta), \quad \text{in which } f = \frac{d^k}{dt^k}(1-t^2)^k,$$

is harmonic and of homogeneous degree k. Their nodal sets consists of parallel slides to the xy-plane, whose latitudes correspond to the zeros of the Legendre function f.

However, there are the analogs to the ones in $\mathbb{R}^2$, for which the nodal sets are easy to picture:

$$\Re(x+iy)^k,\ \Im(y+iy)^k,\ \Re(y+iz)^k,\ \Im(y+iz)^k,\ \Re(z+ix)^k,\ \Im(z+ix)^k.$$

For example, the nodal set of $\Re(x+iy)^k$ or $\Im(y+iy)^k$ is a union of k planes that intersect at the z-axis at equal angles π/k. Hence, $\mathcal{H}^2(\mathcal{N}(P) \cap B^3(0,r)) = k \cdot \text{Area}(B^2(0,r)) = \pi r^2 k$ for P as one of the above.

Example ($\mathbb{H}_k$ in $\mathbb{R}^n$). In $\mathbb{R}^n$,

$$\dim \mathbb{H}_k = O_n\left(k^{n-2}\right) \quad \text{as } k \to \infty.$$

In this space of large dimension, most homogeneous harmonic polynomials have complicated nodal sets, which can not be computed explicitly. However, again the analogs to the ones in $\mathbb{R}^2$ have easy nodal portrait:

$$\Re(x_j + ix_l)^k \quad \text{and} \quad \Im(x_j + ix_l)^k, \quad j \neq l. \tag{3.1}$$

Indeed, the nodal set of $\Re(x_j+ix_l)^k$ or $\Im(x_j+ix_l)^k$ is a union of k hyperplanes that intersect at the $\{x_j = x_l = 0\}$ at equal angles π/k. Hence, $\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B^n(0,r)) = k \cdot \mathcal{H}^{n-1}(B^{n-1}(0,r)) = \alpha_{n-1} r^{n-1} k$ for P_k in (3.1).

Therefore, we verify that the nodal size in $B^n(x,r)$ of the homogeneous harmonic polynomials in (3.1) are controlled by their doubling index:

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B^n(0,r)) = \alpha_{n-1} r^{n-1} k = \alpha_{n-1} r^{n-1} N_P(0,r).$$

For general homogeneous harmonic polynomials, it is impossible to verify the nodal size directly, especially as we are interested in the limit as $k \to \infty$. We next use the powerful integral geometric formula to establish a general theorem.

THEOREM 3.2 (Integral geometric formula). *Let $\mathcal{P}_j : \mathbb{R}^n \to \mathbb{R}$ be the projection from $\mathbb{R}^n$ to the one deleting the j-th variable, i.e.,*

$$\mathcal{P}_j(x_1, ..., x_n) = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n).$$

Suppose that E is a smooth $(n-1)$-dim *hypersurface in $\mathbb{R}^n$. Write*

$$a_j = \int_{\mathbb{R}^{n-1}} \mathcal{H}^0(E \cap \mathcal{P}_j^{-1}(y))\, dy.$$

Then

$$\left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} \leq \mathcal{H}^{n-1}(E) \leq \sum_{j=1}^n a_j.$$

See Han-Lin [**HL2**, Theorem 1.2.10] for more background of the integral geometric formula. In particular, it has a clear geometric interpretation: To estimate the size of an $(n-1)$-dim object in $\mathbb{R}^n$, we only need to examine the intercepts of this set with all straight lines parallel to axis.

THEOREM 3.3 (Upper bound of the nodal size of homogeneous harmonic polynomials). *For any $P \in \mathbb{H}^k$ in $B(0,r)$, we have that*

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B(0,r)) \leq C \cdot r^{n-1} k = C \cdot r^{n-1} N_P(0,r),$$

in which $C = C(n) = n\alpha_{n-1}$.

That is, Conjecture 3.1 holds for homogeneous harmonic polynomials in $B(0,r)$.

PROOF. Fix $y = (y_1, ..., y_{j-1}, y_{j+1}, ..., y_n) \in \mathbb{R}^{n-1}$. If

$$(y_1, ..., y_{j-1}, t, y_{j+1}, ..., y) \in \mathcal{N}(P) \cap \mathcal{P}_j^{-1}(y).$$

then

$$P_y(t) := P(y_1, ..., y_{j-1}, t, y_{j+1}, ..., y) = 0.$$

Observe that $P_y(t)$ is a single-variable polynomial of degree at most k since $P \in \mathbb{H}_k$. Therefore, by the fundamental theorem of algebra,

$$\mathcal{H}^0(\mathcal{N}(P) \cap \mathcal{P}_j^{-1}(y)) = \#\{t : P_y(t) = 0\} \leq k.$$

Hence, in $B(0,r)$,

$$a_j = \int_{\mathbb{R}^{n-1}} \mathcal{H}^0(\mathcal{N}(P) \cap \mathcal{P}_j^{-1}(y))\, dy \leq \int_{B^{n-1}(0,r)} k\, dy = \alpha_{n-1} r^{n-1} k.$$

Applying the integral geometric formula in Theorem 3.2, this theorem follows. □

Remark.

- The first half of the inequality in Theorem 3.3 in fact holds for all polynomials, i.e. the nodal size of all polynomials of degree k in $B(0,r)$ is bounded by $Cr^{n-1}k$. See Problem 3-2. Therefore, the additional message of the theorem is that for homogeneous harmonic polynomials, the degree equals the doubling index in $B(0,r)$.
- The constant $C = n\alpha_{n-1}$ (for example, $C_2 = 4$) in Theorem 3.3 is unlikely sharp. In fact, Guth [**Gu**, Example 2 on Page 1794] conjectured that the sharp constant is α_n, which is achieved by $\Re(x_j + ix_k)^k$ and $\Im(x_j + ix_k)^k$ for $j \neq k$ (for example, $r^k \sin(k\theta)$ and $r^k \cos(k\theta)$ in $\mathbb{R}^2$).
- To understand the potential "overuse" of integral geometric formula, we apply Theorem 3.3 to $r^k \sin(k\theta)$ and $r^k \cos(k\theta)$. One sees that the number of zeros of $P_y(t)$ rarely saturates the degree k. However, the difficulty to prove Guth's conjecture is to design a more efficient integral geometric formula, potentially involving different "test curves" other than the lines parallel to axes in Theorem 3.2.

$\boxed{\textbf{Problems}}$.

3-1. Prove that

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B(0,r)) \leq n\alpha_{n-1} \cdot r^{n-1}k$$

for all polynomials P of degree k. (Hint: Repeat the proof of Theorem 3.3.)

3-2. Find examples of polynomials P such that

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B(0,1)) = \alpha_{n-1} \cdot r^{n-1}k.$$

3-3. Find examples of polynomials P such that the nodal set $\mathcal{N}(P) = \emptyset$.

3.2. Lower bound of the nodal size: Take I

Theorem 3.3 does not provide any lower bound on the nodal size of the polynomials, even though the integral geometric formula in Theorem 3.2 contains a lower bound. This is because the fundamental theorem of algebra does not give useful lower bound of number of zeros of polynomials in the real domain. In fact, a polynomial may have no zeros in the real domain. (See Problem 3-3.) This reflects the difficulty of the following conjecture.

Conjecture 3.4 (Lower bound of the nodal size of harmonic functions). *There is a positive constant $c = c(n)$ such that*

$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x_0,r)) \geq c \cdot r^{n-1}$$

for all harmonic functions u in $B(x_0,r)$ with $u(x_0) = 0$.

See Nadirashvili [**N**] for the original text.

Remark. The lower bound may be invalid without the condition that the function vanishes at some point. For example, $u = 1$ is harmonic but has no zeros. In fact, given any harmonic function u in a ball B, let $M = \sup_B |u|$. Then $v(x) := u(x) + M + 1$ is harmonic but has no zeros in B.

In this section, we prove Conjecture 3.4 in $\mathbb{R}^2$. It is a direct consequence of the maximal principle of the harmonic functions.

THEOREM 3.5 (Lower bound of the nodal size of harmonic functions in $\mathbb{R}^2$). *We have that*

$$\mathcal{H}^1(\mathcal{N}(u) \cap B(x_0, r)) \geq r$$

for all harmonic functions u in $B(x_0, r)$ with $u(x_0) = 0$.

PROOF. Without loss of generality, we assume that $x_0 = (0, 0)$. (If not, then consider the harmonic function $v(x) = u(x - x_0)$.)

Since $u(x_0) = 0$, the vanishing order $V_u(x_0) := d \geq 0$. Since u is analytic by Theorem 2.13 and $u(x_0) = 0$, we have that

$$u(x) = \sum_{j=1}^{\infty} P_j(x) = P_{d+1}(x) + R(x),$$

in which P_{d+1} is a homogeneous polynomial of degree $d + 1$ and $R(x) = O\left(|x|^{d+2}\right)$ as $x \to 0$. That is, the Taylor expansion of u at x_0 begins from the $(d + 1)$-th order term. Because u is harmonic, P_{d+1} is also harmonic.

From Section 3.1, the nodal set of P_{d+1} in $B(x_0, r)$ is a union of d diameters which intersect at x_0 with equal angles of $\pi/(d+1)$. Therefore, the nodal set of u near x_0 is a union of d curves (which also intersect at x_0 with equal angles of $\pi/(d + 1)$.)

Choose l as one of the nodal curves that passes through x_0.

Case 1. If l forms a closed loop, then denote the interior of this loop by Ω. Hence, $u = 0$ on $\partial\Omega = l$. By the maximal principles of the harmonic functions in Theorem 2.3, $u = 0$ in Ω. (In fact, since u is analytic, $u = 0$ in $B(x_0, r)$.) The lower bound readily follows. Indeed,

$$\mathcal{H}^1(\mathcal{N}(u) \cap B(x_0, r)) \geq \mathcal{H}^1(\Omega) \geq \infty,$$

since Ω is 2-dim.

Case 2. If l extends to the boundary $\partial B(x_0, r) = S(x_0, r)$, then l connects x_0 with $S(x_0, r)$. Hence,

$$\mathcal{H}^1(\mathcal{N}(u) \cap B(x_0, r)) \geq \mathcal{H}^1(l) \geq \operatorname{dist}(x_0, S(x_0, r)) = r.$$

$\square$

Remark.

- The above argument of the lower bound breaks down in $\mathbb{R}^n$ for $n \geq 3$. For example, let $x_0 = (0, 0, 0)$ and u be harmonic in $B(x_0, r) \subset \mathbb{R}^3$ with $u(x_0) = 0$. By the maximal principle, any surface s that contains x_0 can not be closed in $B(x_0, r)$ unless $u = 0$ in $B(x_0, r)$ and we are done. In the case when s extends to the boundary $S(x_0, r)$, there is no useful lower bound of the surface area $\mathcal{H}^2(s)$. Indeed, one can have a very narrow surface with arbitrarily small girth and therefore arbitrarily small surface area that contains x_0 and extends to $S(x_0, r)$. (Imagine a very thin finger that touches the center from outside of the ball.) Hence, the lower bound in Conjecture 3.4 does not follow directly from the maximal principle.
- Even for harmonic polynomials in $\mathbb{R}^n$, the lower bound in Conjecture 3.4 is challenging. See Appendix A for more discussion.

- Conjecture 3.4 in $\mathbb{R}^n$ for all $n \geq 3$ has recently been solved by Logunov [**Lo2**]. In fact, Logunov established the lower bound for solutions to general elliptic PDEs (including the harmonic functions). The argument in [**Lo2**] is related to the upper bound discussed in this note but is beyond the scope of our coverage.

Recall in Chapter 2 that $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$ is harmonic in $\mathbb{R}^{n+1}$ if ϕ is an eigenfunction in $\mathbb{R}^n$ with eigenvalue $\lambda > 0$. Their nodal sets are also related in a simple fashion. We now present an immediate consequence of Conjecture 3.4 on the lower bound of the nodal size of eigenfunctions in Yau's conjecture 1.1.

Corollary 3.6 (Lower bound of the nodal size of eigenfunctions). *There is a positive constant $c = c(n)$ such that*

$$\mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_0, r)) \geq c \cdot r^{n-1}\lambda^{\frac{1}{2}}$$

for all eigenfunctions ϕ in $B(x_0, r)$, $-\Delta\phi = \lambda\phi$ with λ sufficiently large.

Proof. We know from Corollary 2.11 that the nodal set of ϕ is "$\lambda^{-1/2}$ dense" in $B(x_0, r/2)$. Therefore, one can find a maximal collection of disjoint balls $B(x_j, t) \subset B(x_0, r/2)$ with $t = r/\sqrt{\lambda}$ such that $\phi(x_j) = 0$, $j = 1, ..., N$. Here, from the maximality of the collection,

$$N \geq c \cdot \frac{r^n}{t^n},$$

in which $c = c(n)$ depends only on the dimension.

Let $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$. Fix $B(x_j, t)$. Then $u(x,t)$ is harmonic in $B(x_j, t) \times [-t, t]$ and $u(x_j, 0) = \phi(x_j) = 0$. By Conjecture 3.4,

$$\mathcal{H}^n(\mathcal{N}(u) \cap B(x_j, t) \times [-t, t]) \geq ct^n.$$

So

$$\mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_j, t)) = \frac{1}{2t} \cdot \mathcal{H}^n(\mathcal{N}(u) \cap B(x_j, t) \times [-t, t]) \geq ct^{n-1}.$$

Hence,

$$\begin{aligned}
\mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_0, r)) &\geq \mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_0, r/2)) \\
&\geq \sum_{j=1}^{N} \mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_j, t)) \\
&\geq c \cdot \frac{r^n}{t^n} \cdot t^{n-1} \\
&\geq c \cdot r^n t^{-1} \\
&= c \cdot r^{n-1}\lambda^{\frac{1}{2}}.
\end{aligned}$$

□

3.3. Harmonic polynomials

In Section 3.1, we introduced the homogeneous harmonic polynomials. We showed that in $B(0, r)$, their nodal size is always bounded above by the degree, which equals the doubling index in $B(0, r)$. The natural questions are what happens in general balls if the center is not necessarily the origin and how about the harmonic polynomials (not necessarily homogeneous).

Recall our notation that $z = x + iy$. Let $P_k = \Im(z^k) = r^k \sin(k\theta)$ in the polar coordinates, which is a homogeneous harmonic polynomial. In the ball $B(p, 1)$ centered at $p = (k, 0)$, we have that

$$\mathcal{N}(P_k) \cap B(p, 1) = \{(t, 0) : k - 1 < t < k + 1\},$$

which has size 2. That is, there is only one nodal diameter of P_k in $B(p, 1)$. So the degree is not an appropriate quantity to estimate the nodal size. However, the doubling index of P_k on $B(p, 1)$ is

$$N_{P_k}(p, 1) = \log_2 \left(\frac{\max_{\mathbb{B}(p,2)} |P_k|}{\max_{\mathbb{B}(p,1)} |P_k|} \right) = \log_2 \left(\frac{(k+2)^k}{(k+1)^k} \right) \le \log_2 \left(\left(1 + \frac{1}{k+1} \right)^{k+1} \right) = \log_2 e.$$

Hence, the doubling index remains the appropriate quantity to control the nodal size:

$$\mathcal{H}^1(\mathcal{N}(P_k) \cap B(p, 1)) \lesssim N_{P_k}(p, 1).$$

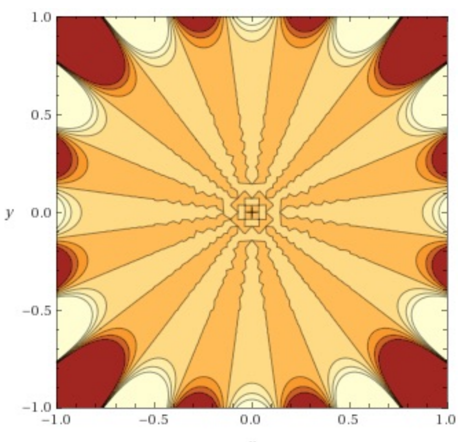
The zero $(0, 0)$ of P_k has vanishing order $k - 1$ but is far away from $B(p, 1)$. Consequently, this zero has no influence on the doubling index (or the nodal size) of P_k in $B(p, 1)$. This example indicates that the doubling index of a polynomial P in $B(p, r)$ only concerns the total vanishing orders of zeros near $B(p, r)$, but not the zeros far away from $B(p, r)$. In the extreme case when $|p| \gg k$, $B(p, r)$ is far away from the zero $(0, 0)$. It may contain no nodal set of P_k and the doubling index at these balls is also bounded by a uniform constant.

In this section, we briefly (and non-rigorously) discuss the harmonic polynomials (not necessarily homogeneous) in $\mathbb{R}^2$ and in $\mathbb{R}^3$ and motivate the relations among the degree, the doubling index, and the nodal size.

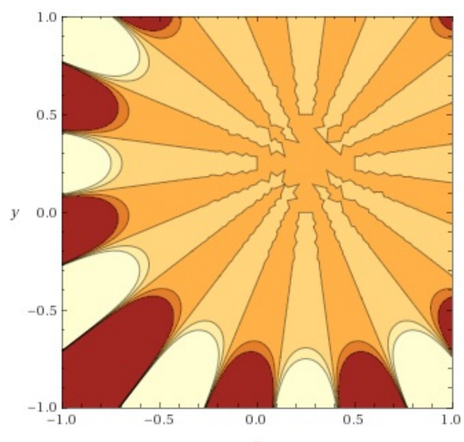
3.3.1. Harmonic polynomials in $\mathbb{R}^2$. Consider the (non-homogeneous) harmonic polynomial in $\mathbb{R}^2$

$$P_k(z) = \Re\left((z - z_1)^{k_1} \cdots (z - z_m)^{k_m} \right),$$

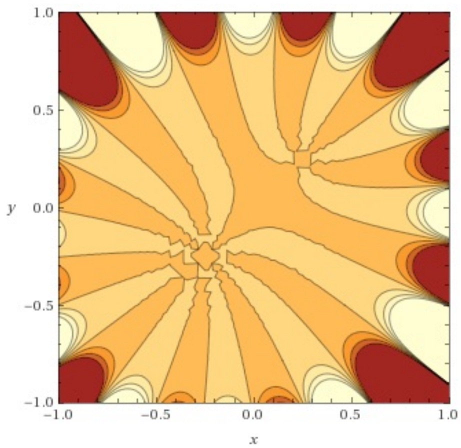
in which $z_1, ..., z_m \in B(0, 1/2)$ and $k_1 + \cdots + k_m = k$. See the following pictures.

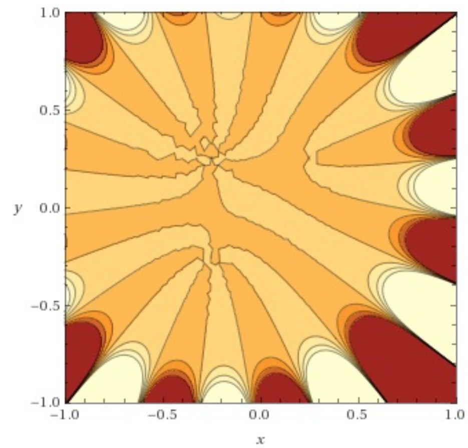


$\Re\left(z^{12}\right)$



$\Re\left(\left(z - \frac{1}{4} - \frac{1}{4}i\right)^{12}\right)$





$\Re\left(\left(z-\frac{1}{4}-\frac{1}{4}i\right)^4\left(z+\frac{1}{4}+\frac{1}{4}i\right)^8\right)$ and $\Re\left(\left(z-\frac{1}{4}-\frac{1}{4}i\right)^2\left(z+\frac{1}{4}+\frac{1}{4}i\right)^4\left(z+\frac{1}{4}-\frac{1}{4}i\right)^6\right)$

The zero z_j in $B(0,1/2)$ with vanishing order k_j-1 accounts for $2k_j$ nodal radials. In total, the nodal size

$$\mathcal{H}^1(\mathcal{N}(P_k)\cap B(0,1))\approx k.$$

To compute the doubling index, observe that

$$|(w-z_1)^{k_1}\cdots(w-z_m)^{k_m}|\approx|w|^k\quad\text{for } |w|\gg\frac{1}{2}.$$

Therefore,

$$N_{P_k}(0,1)=\log_2\left(\frac{\max_{\mathbb{B}(0,2)}|P_k|}{\max_{\mathbb{B}(0,1)}|P_k|}\right)\approx k,$$

which controls the nodal size.

To test the effect of a zero far away from $B(0,1)$ on the doubling index and nodal size, we add a factor $(z-z_0)^{k_0}$ into the polynomial:

$$\Re\left((z-z_1)^{k_1}\cdots(z-z_m)^{k_m}(z-z_0)^{k_0}\right),\quad\text{where } |z_0|\gg 1.$$

Then the degree of this polynomial is $k+k_0$. However, as $|z_0|\gg 1$, $(z-z_0)^{k_0}$ is almost constant near $B(0,1)$. The doubling index and the nodal pattern are both unchanged in $B(0,1)$, i.e., it is still controlled by $k_1+\cdots+k_m$. Summarizing our intuition:

The nodal size of harmonic polynomials P in $B(p,r)$ is bounded above by the doubling index $N_P(p,r)$, which is related to the total vanishing orders of zeros near $B(p,r)$.

Moreover, such intuition leads to our partial answers to the two questions posed in the beginning of this chapter, in the case of harmonic polynomials in $\mathbb{R}^2$:

(1). Monotonicity: $N_P(p,r_1)\lesssim N_P(p,r_2)$ for $r_1\le r_2$, since $B(p,r_2)$ may contain more zeros of the polynomial than $B(p,r_1)$ which results larger doubling index.

(2). Additivity: $N_P(p,R)\gtrsim\sum_j N_P(p_j,r_j)$ for disjoint $B(x_j,r_j)\subset B(x,R)$, since $N_P(p,r)$ accounts for the total vanishing orders of zeros in $B(p,r)$.

3.3.2. Harmonic polynomials in $\mathbb{R}^3$. The harmonic polynomials in $\mathbb{R}^3$ are much more diverse than the ones discussed in $\mathbb{R}^2$. We only mention

$$P_k(x,y,z)=\Re\left((x+iy)^k\right)$$

and demonstrate the intricacy of the additivity of the doubling index. We know that $N_{P_k}(0,r) = k$. In fact, for any point p on the z-axis: $p = (0,0,z)$

$$N_{P_k}(p,r) = \log_2\left(\frac{\max_{\mathbb{B}(p,2r)}|P_k|}{\max_{\mathbb{B}(p,r)}|P_k|}\right) = k.$$

Take disjoint balls $B(p_j,r_j)$ in $B(0,R)$ such that p_j are on the z-axis. Then the additivity of doubling index for these small balls

$$N_u(0,R) \geq \sum_j N_u(p_j,r_j)$$

can never be true.

This example is very special, but indicates that the additivity of the doubling index is highly dependent on the relative positions of the small balls $B(p_j,r_j)$ and large ball $B(0,R)$. In particular, the additivity fails if the small balls $B(p_j,r_j)$ are all on the same straight line.

However, the doubling index of P_k at $B(p,r)$ drops if p is away from z-axis. That is, if $p = (x,y,z)$ such that $|x+iy| = L$, then

$$N_{P_k}(p,r) = \log_2\left(\frac{\max_{\mathbb{B}(p,2r)}|u|}{\max_{\mathbb{B}(p,r)}|u|}\right) = \log_2\left(\frac{|L+2r|^k}{|L+r|^k}\right) \leq k\log_2\left(1+\frac{r}{L}\right),$$

in which

$$\log_2\left(1+\frac{r}{L}\right) \ll 1 \quad \text{if } L \gg r.$$

Hence, if the balls $B(p_j,r_j)$ are scattered away from z-axis, the additivity is possible. In Sections 3.6 and 3.7, we study the (weak) additivity when the small balls are placed according to certain geometric rules.

3.4. Monotonicity of the frequency function

Definition (Frequency function). Let $u \in C^0(\Omega)$. For $\mathbb{B}(x,r) \subset \Omega$, define

$$H_u(x,r) = \int_{S(x,r)} |u(y)|^2\, dS_y.$$

Then the frequency of u in $B(x,r)$ is defined as

$$F_u(x,r) = \frac{rH_u'(x,r)}{2H_u(x,r)}.$$

If there is only one function in question, then we omit the subscript u and write $H(x,r)$ and $F(x,r)$; if the center x is also fixed in the discussion, then we simply write $H(r)$ and $F(r)$.

Remark (Frequency function vs. doubling index). Recall that the doubling index of u on $B(x,r)$ is defined by

$$N_u(x,r) = \log_2\left(\frac{\max_{\mathbb{B}(x,2r)}|u|}{\max_{\mathbb{B}(x,r)}|u|}\right).$$

It is closely related to the frequency function. In fact, we show in Theorem 3.12 that the doubling index and the frequency functions of a harmonic function are indeed comparable. However, concerning the important properties:

(1). Monotonicity: For a fixed center x,

$$\text{is } N_u(x,r) \text{ or } F_u(x,r) \text{ monotone in } r?$$

It is more friendly to work with the frequency function $F_u(x,r)$, since one can differentiate with respect to r. In this section, we establish the crucial monotonicity formula for the frequency function (Theorems 3.7 and 3.8). Since the doubling index is comparable with the frequency function, the results in this section also yield monotonicity of the doubling index in Section 3.5.

(2). Additivity: Let $B(x_j,r_j) \subset B(x,R)$ be disjoint.

$$\text{Is } N_u(x,R) \geq \sum_j N_u(x_j,r_j) \text{ or } F_u(x,R) \geq \sum_j F_u(x_j,r_j)?$$

It is more friendly to work with the doubling index $N_u(x,r)$, since there are direct relations among the supremum of the functions on different balls, for example, $\max_{\mathbb{B}(x_j,r_j)} |u| \leq \max_{\mathbb{B}(x,R)} |u|$ since $B(x_j,r_j) \subset B(x,R)$. From the examples of harmonic polynomials in Section 3.3, we already see that the additivity can not be expected in general. Nevertheless, in Sections 3.6 and 3.7, we establish certain weaker formulations of additivity.

Before proving the monotonicity formulas of the frequency function, we derive some equivalent forms of it. Since

$$H_u(x,r) = \int_{S(x,r)} |u(y)|^2 \, dS_y = r^{n-1} \int_{S(0,1)} |u(x+rz)|^2 \, dS_z,$$

we compute that

$$\begin{aligned} H_u'(x,r) &= (n-1)r^{n-2} \int_{S(0,1)} |u(x+rz)|^2 \, dS_z + 2r^{n-1} \int_{S(0,1)} u(x+rz)\partial_\nu u(x+rz) \, dS_z \\ &= \frac{n-1}{r} \int_{S(x,r)} |u(y)|^2 \, dS_y + 2 \int_{S(x,r)} u(y)\partial_\nu u(y) \, dS_y. \end{aligned}$$

Hence,

$$F_u(x,r) = \frac{rH_u'(x,r)}{2H_u(x,r)} = \frac{n-1}{2} + \frac{r\int_{S(x,r)} u(y)\partial_\nu u(y)\, dS_y}{\int_{S(x,r)} |u(y)|^2 \, dS_y} = \frac{n-1}{2} + \frac{rD_u(x,r)}{H_u(x,r)}, \tag{3.2}$$

in which

$$D_u(x,r) = \int_{S(x,r)} u(y)\partial_\nu u(y) \, dS_y.$$

Remark. The frequency function was initially introduced by Almgren [**A**]. See Han-Lin [**HL2**] for a complete treatment of frequency function and its application to the nodal set estimates of harmonic functions and eigenfunctions. The readers should be warned that the frequency function in [**A, HL2, Li**] is slightly different from (3.2) in this note (which follows Logunov [**Lo1, Lo2**]). Indeed, for harmonic functions, the frequency function in [**A, HL2, Li**] is defined as

$$\frac{rD_u(x,r)}{H_u(x,r)},$$

which differs with (3.2) by a constant $(n-1)/2$. Such difference however has little effect on the results discussed in this note.

We now prove the crucial monotonicity property for the frequency function.

THEOREM 3.7 (Monotonicity of the frequency function). *Let u be harmonic in $B(x_0, R)$. Then $F(r)$ is increasing for $r \in (0, R)$.*

We use the simplified notations $\partial_j = \partial_{x_j}$ and $\partial_{jk} = \partial_{x_j}\partial_{x_k}$ in the proof.

PROOF. Without loss of generality, we assume that $x_0 = 0$. (In the general case, let $w(x) := u(x + x_0)$. Then w is harmonic in $B(0, R)$.) Since (3.2) reads

$$F(r) = \frac{rH'(r)}{2H(r)} = \frac{n-1}{2} + \frac{rD(r)}{H(r)},$$

it suffices to show that

$$F'(r) = \left(\frac{rD(r)}{H(r)}\right)' = \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{rD(r)H'(r)}{H(r)^2} = \frac{rD(r)}{H(r)}\left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right) \geq 0.$$

To this end, we first estimate $H'(r)$: Since

$$H(r) = \int_{S(0,r)} |u(x)|^2 \, dS_x = r^{n-1} \int_{B(0,1)} |u(ry)|^2 \, dS_y,$$

we compute that

$$\begin{aligned} H'(r) &= (n-1)r^{n-2} \int_{S(0,1)} |u(ry)|^2 \, dS_y + 2r^{n-1} \int_{S(0,1)} u(ry)\partial_\nu u(ry) \, dS_y \\ &= \frac{n-1}{r} \int_{S(0,r)} |u(x)|^2 \, dS_x + 2 \int_{S(0,r)} u(x)\partial_\nu u(x) \, dS_x \end{aligned}$$

Hence,

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2\int_{S(0,r)} u(x)\partial_\nu u(x) \, dS_x}{\int_{S(0,r)} |u(x)|^2 \, dS_x}.$$

Next, we estimate $D'(r)$: Since u is harmonic,

$$D(r) = \int_{S(x,r)} u(x)\partial_\nu u(x) \, dS_x = \int_{B(0,r)} |\nabla u(x)|^2 \, dx,$$

by Green's formula in Theorem 1.3. Hence,

$$\begin{aligned} D'(r) &= \left(\int_{B(0,r)} |\nabla u(x)|^2 \, dx\right)' \\ &= \int_{S(0,r)} |\nabla u(x)|^2 \, dS_x \\ &= \frac{1}{r} \int_{S(0,r)} |\nabla u(x)|^2 x \cdot \frac{x}{r} \, dS_x \\ &= \frac{1}{r} \sum_{j=1}^n \int_{S(0,r)} |\nabla u(x)|^2 x_j \frac{x_j}{r} \, dS_x \\ &= \frac{1}{r} \sum_{j=1}^n \int_{S(0,r)} |\nabla u(x)|^2 x_j \nu_j(x) \, dS_x, \end{aligned}$$

in which for fixed $j = 1, ..., n$, we have that

$$\int_{S(0,r)} |\nabla u(x)|^2 x_j \nu_j(x) \, dS_x$$

$$\begin{aligned}
&= \int_{B(0,r)} \partial_j \left(|\nabla u(x)|^2 x_j\right)\, dx \quad \text{(by fundamental theorem of calculus (1.3))}\\
&= \int_{B(0,r)} |\nabla u(x)|^2\, dx + 2\sum_{k=1}^n \int_{B(0,r)} x_j \partial_k u(x) \partial_{jk} u(x)\, dx\\
&= \int_{B(0,r)} |\nabla u(x)|^2\, dx - 2\sum_{k=1}^n \int_{B(0,r)} \partial_k (x_j \partial_k u(x)) \partial_j u(x)\, dx\\
&\quad +2\sum_{k=1}^n \int_{S(0,r)} x_j \partial_k u(x) \partial_j u(x) \nu_k(x)\, dx \quad \text{(by integration by parts (1.4))}\\
&= \int_{B(0,r)} |\nabla u(x)|^2\, dx - 2\int_{B(0,r)} (\partial_j u(x))^2\, dx - 2\sum_{k=1}^n \int_{B(0,r)} x_j \partial_{kk} u(x) \partial_j u(x)\, dx\\
&\quad +2r\sum_{k=1}^n \int_{S(0,r)} \nu_j \partial_k u(x) \partial_j u(x) \nu_k(x)\, dx \quad \left(\text{since } \nu_j = \frac{x_j}{r}\right)\\
&= \int_{B(0,r)} |\nabla u(x)|^2\, dx - 2\int_{B(0,r)} (\partial_j u(x))^2\, dx + 2r\int_{S(0,r)} \nu_j \partial_j u(x) \partial_\nu u(x)\, dx \quad \text{(since } \Delta u = 0).
\end{aligned}$$

Summing over k, we get

$$\begin{aligned}
&D'(r)\\
&= \frac{1}{r}\sum_{j=1}^n \int_{S(0,r)} |\nabla u(x)|^2 x_j \nu_j(x)\, dS_x\\
&= \frac{1}{r}\sum_{j=1}^n \left(\int_{B(0,r)} |\nabla u(x)|^2\, dx - 2\int_{B(0,r)} (\partial_j u(x))^2\, dx + 2r\int_{S(0,r)} \nu_j \partial_j u(x) \partial_\nu u(x)\, dx\right)\\
&= \frac{n}{r}\int_{B(0,r)} |\nabla u(x)|^2\, dx - 2\int_{B(0,r)} \sum_{j=1}^n (\partial_j u(x))^2\, dx + 2\int_{S(0,r)} \partial_\nu u(x) \sum_{j=1}^n \nu_j \partial_j u(x)\, dx\\
&= \frac{n-2}{r}\int_{B(0,r)} |\nabla u(x)|^2\, dx + 2\int_{S(0,r)} |\partial_\nu u(x)|^2\, dx\\
&= \frac{n-2}{r} D(r) + 2\int_{S(0,r)} |\partial_\nu u(x)|^2\, dx.
\end{aligned}$$

Recall the Green formula in Theorem 1.3 that

$$D(r) = \int_{B(0,r)} |\nabla u(x)|^2\, dx = \int_{S(0,r)} u(x) \partial_\nu u(x)\, dx.$$

Hence,

$$\frac{D'(r)}{D(r)} = \frac{n-2}{r} + \frac{2\int_{S(0,r)} |\partial_\nu u(x)|^2\, dx}{\int_{S(0,r)} u(x)\partial_\nu u(x)\, dx}.$$

Putting them together:

$$F'(r) \quad = \quad \frac{rD(r)}{H(r)}\left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right)$$

$$
\begin{aligned}
&= 2N(r)\left(\frac{\int_{S(0,r)}|\partial_\nu u(x)|^2\,dS_x}{\int_{S(0,r)}u(x)\partial_\nu u(x)\,dS_x}-\frac{\int_{S(0,r)}u(x)\partial_\nu u(x)\,dS_x}{\int_{S(0,r)}|u(x)|^2\,dS_x}\right)\\
&\geq 0,
\end{aligned}
$$

since

$$\left(\int_{S(0,r)}u(x)\partial_\nu u(x)\,dx\right)^2\leq\int_{S(0,r)}|\partial_\nu u(x)|^2\,dx\cdot\int_{S(0,r)}|u(x)|^2\,dS_x$$

by the Cauchy-Schwarz inequality. □

An immediate consequence is the following monotonicity formula.

THEOREM 3.8 (Monotonicity formula of the frequency function). *Let u be harmonic in $B(x,R)$. Then for any $0<r_1<r_2<R$, we have that*

$$H(r_2)=H(r_1)\exp\left(\int_{r_1}^{r_2}\frac{2F(r)}{r}\,dr\right).$$

By Theorem 3.7,

$$\left(\frac{r_2}{r_1}\right)^{2F(r_1)}\leq\frac{H(r_2)}{H(r_1)}\leq\left(\frac{r_2}{r_1}\right)^{2F(r_2)},$$

that is,

$$\left(\frac{r_2}{r_1}\right)^{2F(r_1)}\leq\frac{\int_{S(x,r_2)}|u|^2}{\int_{S(x,r_1)}|u|^2}\leq\left(\frac{r_2}{r_1}\right)^{2F(r_2)}.$$

Furthermore,

$$\left(\frac{r_2}{r_1}\right)^{2F(r_1)+1}\leq\frac{\int_{B(x,r_2)}|u|^2}{\int_{B(x,r_1)}|u|^2}\leq\left(\frac{r_2}{r_1}\right)^{2F(r_2)+1}.$$

PROOF. Since $F(r)=rH'(r)/(2H(r))$, $H'(r)/H(r)=rF(r)/2$. Integrating both sides, we have that

$$\log H(r_2)-\log H(r_1)=\int_{r_1}^{r_2}\frac{2F(r)}{r}\,dr.$$

The equation in the theorem therefore follows. By Theorem 3.7, $F(r_1)\leq F(r)\leq F(r_2)$ for all $r\in[r_1,r_2]$. Hence,

$$2F(r_1)\log\left(\frac{r_2}{r_1}\right)\leq\int_{r_1}^{r_2}\frac{2F(r)}{r}\,dr\leq 2F(r_2)\log\left(\frac{r_2}{r_1}\right).$$

The inequality about $H(r)$ in the theorem follows. To prove the last inequalities, notice that

$$
\begin{aligned}
\int_{B(x,r_2)}|u|^2 &= \int_0^{r_2}\int_{S(x,r)}|u|^2\,dSdr\\
&= \frac{r_2}{r_1}\int_0^{r_1}\int_{S(x,tr_2/r_1)}|u|^2\,dSdt\\
&\leq \frac{r_2}{r_1}\int_0^{r_1}\left(\frac{r_2}{r_1}\right)^{2F(x,r_2)}\int_{S(x,t)}|u|^2\,dSdt\\
&\leq \left(\frac{r_2}{r_1}\right)^{2F(x,r_2)+1}\int_0^{r_1}\int_{S(x,t)}|u|^2\,dSdt
\end{aligned}
$$

$$= \left(\frac{r_2}{r_1}\right)^{2F(x,r_2)+1} \int_{B(x,r_1)} |u|^2,$$

which implies the upper bound in the last inequalities. The lower bound can be proved similarly. □

After proving the monotonicity of the frequency function $F_u(x,r)$ for fixed center, a natural question involves the frequency function at different centers. To this end, we prove the following theorem.

THEOREM 3.9. *Let $K \geq 3$ and u be harmonic in $B(x, Kr)$. Then for any $y \in B(x,r)$ and $2 \leq t \leq K-1$, we have that*

$$F_u(y,tr) \leq \frac{t\log K}{K-1} F_u(x,Kr) + \frac{t\log K}{2(K-1)} - \frac{1}{2}.$$

If $0 < t \leq 2$, then we apply the monotonicity of the frequency function and get

$$F_u(y,tr) \leq F_u(y,2r) \leq \frac{2\log K}{K-1} F_u(x,Kr) + \frac{\log K}{K-1} - \frac{1}{2} \leq c_1 F_u(x,Kr) + c_2,$$

in which $c_1 = c_1(K)$ and $c_2 = c_2(K)$ are positive constants that depend only on K.

PROOF. Since $y \in B(x,r)$ and $2 \leq t \leq K-1$,

$$B(x,r) \subset B(y,tr) \subset B(y,(K-1)r) \subset B(x,Kr).$$

Using Theorem 3.8, we have that

$$\begin{aligned} \int_{B(y,(K-1)r)} |u|^2 &\leq \int_{B(x,Kr)} |u|^2 \\ &\leq K^{2F_u(x,Kr)+1} \int_{B(x,r)} |u|^2 \\ &\leq K^{2F_u(x,Kr)+1} \int_{B(y,tr)} |u|^2. \end{aligned}$$

Therefore,

$$\frac{\int_{B(y,(K-1)r)} |u|^2}{\int_{B(y,tr)} |u|^2} \leq K^{2F_u(x,Kr)+1}.$$

Using Theorem 3.8 again,

$$\frac{\int_{B(y,(K-1)r)} |u|^2}{\int_{B(y,tr)} |u|^2} \geq \left(\frac{K-1}{t}\right)^{2F_u(y,tr)+1}.$$

Hence,

$$\left(\frac{K-1}{t}\right)^{2F_u(y,2r)+1} \leq K^{2F_u(x,Kr)+1},$$

which implies that

$$F_u(x,tr) \leq \frac{t\log K}{K-1} F_u(x,Kr) + \frac{t\log K}{2(K-1)} - \frac{1}{2}.$$

□

3.5. Monotonicity of the doubling index

For general harmonic functions, we first note that if u is harmonic in a neighborhood of x, then $N_u(x,r) \to d+1$ as $r \to 0$ if $V_u(x) = d$, i.e., u vanishes at x of order d. Indeed, let u have vanishing order d at x. Then $\partial^\alpha u(x) = 0$ for all $|\alpha| \le d$ and there is α with $|\alpha| = d+1$ such that $\partial^\alpha u(x) \neq 0$. By the Taylor expansion,

$$u(y) = \sum_{|\alpha|=d+1} \frac{\partial^\alpha u(x)(y-x)^\alpha}{\alpha!} + R(y-x) := P(y-x) + R(y-x),$$

in which the summation defines a homogeneous polynomial P of degree $d+1$ and the remainder $R(y-x) = O\left(|y-x|^{d+2}\right)$. One then deduces that $N_u(x,r) \to N_P(x,r) = d+1$ as $r \to 0$.

In this section, we prove the monotonicity properties for the doubling index, with the help of frequency function. We first see that the frequency function equals the degree (and therefore the doubling index) module a constant $(n-1)/2$ for homogeneous harmonic polynomials in $B(0,r)$.

THEOREM 3.10. *For each $P \in \mathbb{H}_k$ in $\mathbb{R}^n$, $F_P(0,r) = k + (n-1)/2 = N_P(0,r) + (n-1)/2$ for all $r > 0$.*

PROOF. Write in the spherical coordinates:

$$P(r,\theta) = r^k \phi(\theta), \quad r \in [0,\infty),\ \theta \in S(0,1).$$

Notice that $\partial_\nu = \partial_r$ on $S(0,r)$. Then

$$\partial_\nu P(r,\theta) = \partial_r P(r,\theta) = k r^{k-1}\phi(\theta).$$

Then

$$\int_{S(0,r)} P \partial_\nu P \, dS = \int_{S(0,r)} r^k \phi(\theta) \cdot k r^{k-1} \phi(\theta)\, dS = \frac{k}{r} \int_{S(0,r)} |P|^2 \, dS.$$

Hence,

$$F_P(0,r) = \frac{n-1}{2} + \frac{r \int_{S(0,r)} P\partial_\nu P \, dS}{\int_{S(0,r)} |P|^2 \, dS} = k + \frac{n-1}{2}.$$

□

We next prove a similar (but less precise) comparison between the frequency function and the doubling index for all harmonic functions. First we need a lemma.

LEMMA 3.11. *Let $\delta > 0$. Then*

$$\sup_{B(x,r)} w \le \max_{\mathbb{B}(x,r)} w \le \left(\frac{1+\delta}{\delta}\right)^n \fint_{S(x,(1+\delta)r)} w \, dS$$

for all subharmonic and non-negative functions w in $\mathbb{B}(x,(1+\delta)r)$.

PROOF. By Problem 2-2,

$$w(x) \le \fint_{B(x,r)} w(y)\, dy.$$

In fact, we have that

$$\fint_{S(x,r)} w(y)\, dS_y = \frac{1}{n\alpha_n r^{n-1}} \int_{S(x,r)} w(y)\, dS_y \quad \text{is increasing in } r.$$

Hence,

$$\int_{B(x,r)} w(y)\, dy \;=\; \int_0^r \left(\int_{S(x,t)} w(y)\, dS_y \right) dt$$

$$
\begin{aligned}
&= \int_0^r \left(n\alpha_n t^{n-1} \fint_{S(x,t)} w(y)\, dS_y \right) dt \\
&\le \fint_{S(x,r)} w(y)\, dS_y \int_0^r n\alpha_n t^{n-1}\, dt \\
&\le \alpha_n r^n \fint_{S(x,r)} w(y)\, dS_y \\
&= \frac{r}{n} \int_{S(x,r)} w(y)\, dS_y.
\end{aligned}
$$

For any $y \in \mathbb{B}(x,r)$, $\mathbb{B}(y,\delta r) \subset \mathbb{B}(x,(1+\delta)r)$. Since w is non-negative,

$$
\begin{aligned}
w(y) &\le \fint_{\mathbb{B}(y,\delta r)} w(z)\, dz \\
&\le \frac{1}{\alpha_n \delta^n r^n} \int_{\mathbb{B}(y,\delta r)} w(z)\, dz \\
&= \frac{1}{\alpha_n \delta^n r^n} \int_{\mathbb{B}(x,(1+\delta)r)} w(z)\, dz \\
&\le \frac{1+\delta}{n\alpha_n \delta^n r^{n-1}} \int_{S(x,(1+\delta)r)} w(z)\, dz \\
&= \left(\frac{1+\delta}{\delta}\right)^n \fint_{S(x,(1+\delta)r)} w(z)\, dz.
\end{aligned}
$$

$\square$

Theorem 3.12 (Comparison between the doubling index and the frequency function of harmonic functions). *Let $\varepsilon > 0$ be small. Then*

$$
\begin{aligned}
(1-\log_2(1+\varepsilon))F_u(x,(1+\varepsilon)r) - \frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right) - \frac{n-1}{2} &\le N_u(x,r) \\
&\le (1+\log_2(1+\varepsilon))F_u(x,2(1+\varepsilon)r) + \frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right) - \frac{n-1}{2}
\end{aligned}
$$

for all harmonic functions u in $\mathbb{B}(x,2(1+\varepsilon)r)$. In particular, observe that $\log_2(1+\varepsilon) < 10\varepsilon$. *Denote*

$$
C = C(\varepsilon,n) = \frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right) + \frac{n-1}{2}.
$$

Then

$$
(1-10\varepsilon)F_u(x,(1+\varepsilon)r) - C \le N_u(x,r) \le (1+10\varepsilon)F_u(x,2(1+\varepsilon)r) + C.
$$

Proof of Theorem 3.12. By Problem 2-1, $|u|^2$ is subharmonic. By the Lemma 3.11,

$$
\max_{\mathbb{B}(x,2r)} |u|^2 \le \left(\frac{1+\varepsilon}{\varepsilon}\right)^n \fint_{S(x,2(1+\varepsilon)r)} |u|^2\, dS = \frac{1+\varepsilon}{\varepsilon^n} \cdot \frac{H_u(x,2(1+\varepsilon)r)}{n\alpha_n 2^{n-1} r^{n-1}}.
$$

On the other hand, since $S(x,r) \subset \mathbb{B}(x,r)$,

$$
\max_{\mathbb{B}(x,r)} |u|^2 \ge \sup_{S(x,r)} |u|^2 \ge \fint_{S(x,r)} |u(y)|^2\, dy = \frac{H_u(x,r)}{n\alpha_n r^{n-1}}.
$$

Using the monotonicity formula in Theorem 3.8,

$$\frac{H_u(x,2(1+\varepsilon)r)}{H_u(x,r)} \le (2(1+\varepsilon))^{2F_u(x,2(1+\varepsilon)r)}.$$

Therefore,

$$\begin{aligned}
N_u(x,r) &= \frac{1}{2}\log_2\left(\frac{\max_{\mathbb{B}(x,2r)}|u|^2}{\max_{\mathbb{B}(x,r)}|u|^2}\right)\\
&\le \frac{1}{2}\log_2\left(\frac{H_u(x,2(1+\varepsilon)r)}{H_u(x,r)}\cdot\frac{(1+\varepsilon)}{2^{n-1}\varepsilon^n}\right)\\
&\le \frac{1}{2}\log_2\left(\frac{H_u(x,2(1+\varepsilon)r)}{H_u(x,r)}\right)+\frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right)-\frac{n-1}{2}\\
&\le \log_2(2(1+\varepsilon))F_u(x,2(1+\varepsilon)r)+\frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right)-\frac{n-1}{2}\\
&\le (1+\log_2(1+\varepsilon))F_u(x,2(1+\varepsilon)r)+\frac{1}{2}\log_2\left(\frac{1+\varepsilon}{\varepsilon^n}\right)-\frac{n-1}{2}.
\end{aligned}$$

To obtain the opposite estimate of N_u and F_u, see Problem 3-4.

□

An immediate consequence is the monotonicity property of the doubling index:

Theorem 3.13 (Monotonicity of the doubling index). *Let $\varepsilon > 0$. There is constant $C = C(\varepsilon, n)$ such that if $r_2 \ge 2r_1 > 0$, then*

$$N_u(x,r_2) \ge (1-\varepsilon)N_u(x,r_1) - C$$

for all harmonic functions u in $\mathbb{B}(x,2r_2)$.

This theorem is obviously weaker than the monotonicity of the frequency function in Theorem 3.7.

Proof. By Theorem 3.7, there is constant $C_1 = C_1(\varepsilon_1, n) > 0$ such that

$$(1-10\varepsilon_1)F_u(x,(1+\varepsilon_1)r) - C_1 \le N_u(x,r) \le (1+10\varepsilon_1)F_u(x,2(1+\varepsilon_1)r) + C_1.$$

By the monotonicity of the frequency function in Theorem 3.7, since $r_2 \ge 2r_1$,

$$\begin{aligned}
N_u(x,r_2) &\ge (1-10\varepsilon_1)F_u(x,(1+\varepsilon_1)r_2) - C_1\\
&\ge (1-10\varepsilon_1)F_u(x,2(1+\varepsilon_1)r_1) - C_1\\
&\ge \frac{1-10\varepsilon_1}{1+10\varepsilon_1}N_u(x,r_1) - \frac{2C_1}{1+10\varepsilon_1}\\
&\ge (1-\varepsilon)N_u(x,r_1) - C,
\end{aligned}$$

by setting $\varepsilon_1 = \varepsilon/40$ and $C = 2C_1/(1+10\varepsilon_1)$. □

Next we prove the monotonicity formula for the doubling index that is similar to Theorem 3.8 for the frequency function.

Theorem 3.14 (Monotonicity formula of the doubling index). *Let $\varepsilon > 0$ be small. Then there exists $C = C(\varepsilon, n) \to \infty$ depending only on $\varepsilon \to 0$ (and the dimension n) such that*

$$t^{(1-\varepsilon)N_u(x,r)-C} \le \frac{\max_{\mathbb{B}(x,tr)}|u|}{\max_{\mathbb{B}(x,r)}|u|} \le t^{(1+\varepsilon)N_u(x,tr)+C}$$

for all harmonic functions u in Ω and $B(x,tr) \subset \Omega$. In addition, there is $N_0 = N_0(\varepsilon, n)$ such that if $N_u(x,r) \geq N_0$, then

$$t^{(1-\varepsilon)N_u(x,r)} \leq \frac{\max_{\mathbb{B}(x,tr)} |u|}{\max_{\mathbb{B}(x,r)} |u|} \leq t^{(1+\varepsilon)N_u(x,tr)}.$$

Proof. We only prove the lower bound, the upper bound can be proved similarly. We can assume that $t > 2^{1/(1-\varepsilon)}$. Otherwise $2 \leq t \leq 2^{1/(1-\varepsilon)}$ so

$$\max_{\mathbb{B}(x,tr)} |u| \geq \max_{\mathbb{B}(x,2r)} |u| \geq 2^{N_u(x,r)} \max_{\mathbb{B}(x,r)} |u| \geq t^{(1-\varepsilon)N_u(x,r)} \max_{\mathbb{B}(x,r)} |u|,$$

which implies the lower bound.

First,

$$\max_{\mathbb{B}(x,tr)} |u|^2 \geq \sup_{S(x,tr)} |u|^2 \geq \fint_{S(x,tr)} |u(y)|^2 \, dS_y = \frac{H_u(x,tr)}{n\alpha_n t^{n-1} r^{n-1}}. \tag{3.3}$$

Since $t > 2^{1/(1-\varepsilon)}$, $(1-\varepsilon)tr > 2(1+\varepsilon)r$. By the monotonicity formula for the frequency function in Theorem 3.8,

$$\begin{aligned} H_u(x,tr) &\geq \left(\frac{t}{2(1+\varepsilon)}\right)^{2F_u(x,2(1+\varepsilon)r)} \cdot H_u(x, 2(1+\varepsilon)r) \\ &\geq \left(\frac{t}{2(1+\varepsilon)}\right)^{\frac{2N_u(x,r)}{1+10\varepsilon} - \frac{2C}{1+10\varepsilon}} \cdot H_u(x, 2(1+\varepsilon)r), \end{aligned}$$

in which we use Theorem 3.12 that

$$(1+10\varepsilon)F_u(x, 2(1+\varepsilon)r) + C \geq N_u(x,r).$$

Second, using the definition of doubling index,

$$\max_{\mathbb{B}(x,2r)} |u| = 2^{N_u(x,r)} \max_{\mathbb{B}(x,r)} |u|,$$

So by Lemma 3.11,

$$\begin{aligned} \max_{\mathbb{B}(x,r)} |u|^2 &= 2^{-2N_u(x,r)} \max_{\mathbb{B}(x,2r)} |u|^2 \\ &\leq 2^{-2N_u(x,r)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^n \fint_{S(x,2(1+\varepsilon)r)} |u(y)|^2 \, dS_y \\ &\leq 2^{-2N_u(x,r)} \cdot \frac{(1+\varepsilon)H_u(x, 2(1+\varepsilon)r)}{n\alpha_n 2^{n-1} \varepsilon^n r^{n-1}}, \end{aligned}$$

which implies that

$$H_u(x, 2(1+\varepsilon)r) \geq \frac{2^{2N_u(x,r)} n\alpha_n 2^{n-1} \varepsilon^n r^{n-1}}{1+\varepsilon} \max_{\mathbb{B}(x,r)} |u|^2.$$

Connecting (3.3) with the subsequent estimates, we have that

$$\begin{aligned} &\max_{\mathbb{B}(x,tr)} |u|^2 \\ \geq\ & \frac{1}{n\alpha_n t^{n-1} r^{n-1}} \cdot H_u(x,tr) \\ \geq\ & \frac{1}{n\alpha_n t^{n-1} r^{n-1}} \cdot \left(\frac{t}{2(1+\varepsilon)}\right)^{\frac{2N_u(x,r)}{1+10\varepsilon} - \frac{2C}{1+10\varepsilon}} \cdot H_u(x, 2(1+\varepsilon)r) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{2^{2N_u(x,r)} n\alpha_n 2^{n-1}\varepsilon^n r^{n-1}}{n\alpha_n(1+\varepsilon)t^{n-1}r^{n-1}} \cdot \left(\frac{t}{2(1+\varepsilon)}\right)^{\frac{2N_u(x,r)}{1+10\varepsilon}-\frac{2C}{1+10\varepsilon}} \cdot \max_{\mathbb{B}(x,r)} |u|^2 \\
&\geq \frac{2^{n-1}\varepsilon^n}{1+\varepsilon} \cdot \left(\frac{t}{1+\varepsilon}\right)^{\frac{2N_u(x,r)}{1+10\varepsilon}-\frac{2C}{1+10\varepsilon}} \cdot t^{-(n-1)} \cdot \max_{\mathbb{B}(x,r)} |u|^2 \\
&= 2^{n-1}\varepsilon^n (1+\varepsilon)^{\frac{2C}{1+10\varepsilon}-1} \cdot \left(\frac{1}{1+\varepsilon}\right)^{\frac{2N_u(x,r)}{1+10\varepsilon}} \cdot t^{\frac{2N_u(x,r)}{1+10\varepsilon}-\frac{2C}{1+10\varepsilon}-(n-1)} \cdot \max_{\mathbb{B}(x,r)} |u|^2.
\end{aligned}
$$

Since $t \geq 2$, one can choose $C_1 = C_1(\varepsilon, n) > 0$ large enough such that

$$2^{n-1}\varepsilon^n(1+\varepsilon)^{\frac{2C}{1+10\varepsilon}-1} \geq 2^{-C_1} \geq t^{-C_1}.$$

In addition, as $\varepsilon \to 0^+$, there is $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0^+$ such that

$$\frac{1}{1+\varepsilon} \geq 2^{-\varepsilon_1} \geq t^{-\varepsilon_1}.$$

Hence,

$$\max_{\mathbb{B}(x,tr)} |u|^2 \leq t^{\frac{2(1-\varepsilon_1)N_u(x,r)}{1+10\varepsilon}-\frac{2C}{1+10\varepsilon}-(n-1)-C_1} \cdot \max_{\mathbb{B}(x,r)} |u|^2,$$

which implies that

$$\frac{\max_{\mathbb{B}(x,tr)} |u|}{\max_{\mathbb{B}(x,r)} |u|} \geq t^{(1-\varepsilon)N_u(x,r)-C}$$

with appropriate (new) ε and C. □

Similar to Theorem 3.9 that compares the frequency function at different centers, we have

THEOREM 3.15. *Let $\delta \in (0,1)$. Then there exist positive constants $K = K(\delta) \geq 1$ and $N_0 = N_0(\delta, n)$ such that if $N_u(x,r) \geq N_0$, then for any $y \in \mathbb{B}(x,r)$, we have that*

$$N_u(y, 2Kr) \geq (1-\delta)N_u(x,r)$$

for all harmonic functions u in $\mathbb{B}(x,(2K+1)r)$.

PROOF. Using Theorem 3.13, there is $C = C(\varepsilon_1, n)$ such that

$$N_u(x,(K+1)r) \geq (1-\varepsilon_1)N_u(x,r) - C,$$

because $K+1 \geq 2$. Since $y \in \mathbb{B}(x,r)$,

$$\mathbb{B}(y,Kr) \subset \mathbb{B}(x,(K+1)r) \quad \text{and} \quad \mathbb{B}(x,(2K-1)r) \subset \mathbb{B}(y,2Kr).$$

Using Theorem 3.14, we have that if $N_u(x,(K+1)r) \geq N_2 = N_2(\varepsilon_2, n)$, then

$$
\begin{aligned}
2^{(1+\varepsilon_2)N_u(y,2Kr)} &\geq \frac{\max_{\mathbb{B}(y,2Kr)} |u|}{\max_{\mathbb{B}(y,Kr)} |u|} \\
&\geq \frac{\max_{\mathbb{B}(x,(2K-1)r)} |u|}{\max_{\mathbb{B}(x,(K+1)r)} |u|} \\
&\geq \left(\frac{2K-1}{K+1}\right)^{(1-\varepsilon_2)N_u(x,(K+1)r)}.
\end{aligned}
$$

See (4) below for the choice of the parameters so the condition $N_u(x,(K+1)r) \geq N_2$ is met. Therefore,

$$N_u(y,2Kr) \geq \frac{1-\varepsilon_2}{1+\varepsilon_2} \cdot \log_2\left(\frac{2K-1}{K+1}\right) \cdot N_u(x,(K+1)r)$$

$$\geq \frac{(1-\varepsilon_2)(1-\varepsilon_1)}{1+\varepsilon_2} \cdot \log_2\left(\frac{2K-1}{K+1}\right) \cdot N_u(x,r) - C$$

Let $\delta > 0$. We now choose the parameters to finish the proof.

(1). Choose $\varepsilon_2 = \varepsilon_2(\delta) > 0$ small enough so

$$\frac{1-\varepsilon_2}{1+\varepsilon_2} \geq 1 - \frac{\delta}{4}.$$

(2). Choose $\varepsilon_1 = \varepsilon_1(\delta) > 0$ small enough so

$$\frac{(1-\varepsilon_2)(1-\varepsilon_1)}{1+\varepsilon_2} \geq 1 - \frac{\delta}{3}.$$

(3). Choose $K = K(\delta) > 0$ large enough so

$$\frac{(1-\varepsilon_2)(1-\varepsilon_1)}{1+\varepsilon_2} \cdot \log_2\left(\frac{2K-1}{K+1}\right) \geq 1 - \frac{\delta}{2}.$$

(4). Choose $N_0 = N_0(\delta, n) > 0$ large enough such that if $N_u(x,r) \geq N_0$, then

$$N_u\left(x, (K+1)r\right) \geq (1-\varepsilon_1)N_u(x,r) - C \geq (1-\varepsilon_1)N_0 - C \geq N_2,$$

and

$$\left(1 - \frac{\delta}{2}\right) N_u(x,r) - C \geq (1-\delta)N_u(x,r).$$

□

Problems.

3-4. Prove the lower bound in Theorem 3.12.

3.6. The simplex lemma

We summarize the main themes of monotoninicity of the doubling index $N_u(x,r)$ in Section 3.5 for harmonic functions:

(i). "$N_u(x,r)$ with fixed center x is increasing in r." – It is put in quotations since this is only morally true. See Theorem 3.13 for details.

(ii). "$N_u(x,r_2)$ and $N_u(x,r_1)$ characterize the growth rate of $|u|$ from $\mathbb{B}(x,r_1)$ to $\mathbb{B}(x,r_2)$ for $r_1 \leq r_2$":

$$\left(\frac{r_2}{r_1}\right)^{N_u(x,r_1)} \leq \frac{\max_{\mathbb{B}(x,r_2)}|u|}{\max_{\mathbb{B}(x,r_1)}|u|} \leq \left(\frac{r_2}{r_1}\right)^{N_u(x,r_2)}.$$

Again, it is put in quotations since the above inequalities are also only morally true. See Theorem 3.14 for details.

In this section, we investigate the additivity property of the doubling index of harmonic functions: Let $B(x_j,r_j) \subset B(x,R)$ be disjoint. Is

$$N_u(x,R) \geq \sum_j N_u(x_j,r_j)?$$

From the example $(x+iy)^k$ in $\mathbb{R}^3$ discussed in Section 3.3, we see that without any geometric constraints, it can fail (for example, if the centers x_j are on the same line). That is, the additivity property is highly dependent on the relative geometric positions of the balls.

In fact, it is extremely challenging to prove any kind of additivity result for harmonic functions. For example, the most fundamental additivity question asks: Given $B(x_j,r_j)$ such that

$N_u(x_j, r_j) = N$, can one find $B(x_0, R)$ such that $N_u(x_0, R) > N$, that is, can the doubling index of $B(x_j, r_j)$ "add up" at $B(x_0, R)$ by a bit?

The simplex lemma in this section provide an answer to this fundamental question. That is, if x_j are the vertices of a non-degenerate simplex (that is, the simplex does not lie in a hyperplane, in particular, x_j are not on the same line,) then the doubling index at the barycenter x_0 of the simplex is strictly greater than the ones at the vertices.

Remark (Intuition of the proof of the simplex lemma). Both of the upper and lower bounds of the growth rate of $|u|$ in terms of the doubling index are provided in Theme (ii) above. This is the starting point of the proof. That is, let $B(x_0, R_1)$ grow to $B(x_0, R_2)$ and $B(x_j, r_1)$ grow to $B(x_j, r_2)$. Then we aim to build a sequence of inequalities so $N_u(x_0, R_2)$ can be bounded by $N_u(x_j, r_1)$ from below:

$$\left(\frac{R_2}{R_1}\right)^{N_u(x_0,R_2)} \geq \frac{\max_{\mathbb{B}(x_0,R_2)}|u|}{\max_{\mathbb{B}(x_0,R_1)}|u|} \geq \frac{\max_{\mathbb{B}(x_j,r_2)}|u|}{\max_{\mathbb{B}(x_j,r_1)}|u|} \geq \left(\frac{r_2}{r_1}\right)^{N_u(x_j,r_1)}.$$

In order for the middle inequality to hold, we need

(I).
$$\max_{\mathbb{B}(x_0,R_2)}|u| \geq \max_{\mathbb{B}(x_j,r_2)}|u|,$$
which amounts to covering balls $B(x_j, r_2)$ at the vertices by the ball $B(x_0, R_2)$ at the barycenter. It is done by Covering Lemma I below.

(II).
$$\max_{\mathbb{B}(x_0,R_1)}|u| \leq \max_{\mathbb{B}(x_j,r_1)}|u|,$$
which amounts to covering the ball $B(x_0, R_1)$ at the barycenter by the balls $B(x_j, r_1)$ at the the vertices. It is done by Covering Lemma II below.

3.6.1. The simplex geometric lemmas.

Definition (Euclidean simplexes). The n-simplex S with vertices $x_1, ..., x_{n+1} \in \mathbb{R}^n$ is defined as the convex hull of $x_1, ..., x_{n+1}$, i.e.,

$$S = \{t_1x_1 + \cdots + t_{n+1}x_{n+1} : t_1 + \cdots + t_{n+1} = 1,\ t_1, ..., t_{n+1} \geq 0\}.$$

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron (i.e., pyramid), and so on. Let S be an n-simplex in $\mathbb{R}^n$.

- Denote $\operatorname{diam}(S)$ the diameter of S, i.e., the maximal distance between a pair of points in S.
- Denote $\operatorname{width}(S)$ the width of S, i.e, the minimal distance between a pair of parallel hyperplanes such that S is contained between them.
- Denote $\omega(S) = \operatorname{width}(S)/\operatorname{diam}(S) \in [0, 1]$ the relative width of S. We say that the simplex is degenerate (or flat) if $\omega(S) = 0$. Abusing the terminology somehow, we say that a simplex is more degenerate (or flat) if ω is smaller.
- Define the barycenter x_0 of S as
$$x_0 = \frac{x_1 + \cdots + x_{n+1}}{n+1}.$$

Keep the above notation throughout this subsection. The following two lemmas show certain covering relations between balls centered at the vertices and balls centered at the barycenter. Later on we apply these covering lemmas to study the relation among the doubling indexes of these balls.

The first covering lemma says that balls with smaller radius can be covered by balls with larger radius, and it therefore straightforward. It is an immediate consequence of the triangle inequality.

LEMMA 3.16 (Covering Lemma I). *Let S be an n-simplex and $r > 0$. Suppose that $t > 0$. Then*

$$\bigcup_{j=1}^{n+1} B(x_j, tr) \subset B(x_0, (1+\delta)tr),$$

for

$$\delta = \frac{\operatorname{diam}(S)}{tr}.$$

We remark that $\delta \to 0$ as $t \to \infty$, that is, as the radius tend to infinity, balls centered at the barycenter and at the vertices are almost the same.

PROOF. Let $j = 1, ..., n$ and $y \in B(x_j, tr)$. By the triangle inequality,

$$|y - x_0| < |y - x_j| + |x_j - x_0| \leq tr + \operatorname{diam}(S) \leq (1+\delta)tr,$$

if

$$\delta \geq \frac{\operatorname{diam}(S)}{tr}.$$

□

The second covering lemma says that balls with larger radius can be covered by balls with smaller radius, and is therefore more complicated than the first lemma. It uses the property of the relative width of a simplex.

LEMMA 3.17 (Covering Lemma II). *Let $a > 0$. Then there exist $c_1 = c_1(a, n) > 0$ and $K = K(a, n) \geq 2/a$ such that*

$$B(x_0, (1+c_1)r) \subset \bigcup_{j=1}^{n+1} B(x_j, r), \quad \text{in which } r = K \cdot \operatorname{diam}(S)$$

for all simplexes S with the relative width $\omega(S) > a$. In addition, $c_1 \to 0^+$ and $K \to \infty$ as $a \to 0$.

Remark.

- In the extreme case when $\omega(S) = 0$ (i.e., S lies on a hyperplane), the above covering property is impossible.
- It is obvious that the above covering property is impossible if K is too small, say $K < 1/100$, so $B(x_j, r)$ are tiny balls centered at the vertices and can never cover a ball centered at the barycenter. The choice of $K \geq 2/a$ is mainly for future application in Section 3.8.

PROOF. See Problems 3-5 and 3-6. □

Using the two covering lemmas and the monotonicity formula in Theorem 3.14, we derive the following simplex lemma. It is a weak form of additivity mentioned above. In particular, it asserts that the doubling index at the barycenter of a simplex must be at least slightly larger than the ones at the vertices, if the simplex is not degeneration. In the view of the example in the beginning of this section, the non-degeneracy of the simplex is necessary, that is, this phenomenon of weak additivity can not be true if the simplex degenerates to a line segment.

Lemma 3.18 (The simplex lemma). *Let $a > 0$ and $K = K(a,n) \geq 2/a$. Suppose that $\omega(S) > a$ and $r = K \cdot \mathrm{diam}(S)$. Then there are positive constants $c = c(a,n)$, $C = C(a,n) \geq K$, and $N_0 = N_0(a,n)$ such that if for each $j = 1, ..., n+1$, $N_u(x_j, r_j) \geq N \geq N_0$ for some $0 < r_j \leq r/2$, then*

$$N_u\left(x_0, C \cdot \mathrm{diam}(S)\right) > (1+c)N$$

for all harmonic functions u.

Remark. We can call the simplex lemma a weak version of additivity of the doubling index: The doubling index at the vertices "add up" (a little bit) to the one at the barycenter.

Proof. By the weak monotonicity of the doubling index in Theorem 3.13, we can assume that $N_u(x_j, r) \geq N$ since $r \geq 2r_j$. Let

$$M = \max_{\cup_{j=1}^{n+1} \mathbb{B}(x_j, r)} |u|.$$

Then $M = \max_{\mathbb{B}(x_j,r)} |u|$ for some $j = 1, ..., n+1$. By the covering lemma in Lemma 3.17, we have that

$$B(x_0, (1+c_1)r) \subset \bigcup_{j=1}^{n+1} B(x_j, r),$$

which implies that

$$\max_{\mathbb{B}(x_0,(1+c_1)r)} |u| \leq M.$$

Let $t, \varepsilon > 0$, whose values will be specified later. By Theorem 3.14, if $N \geq N_0$, then

$$\max_{\mathbb{B}(x_j,tr)} |u| \geq t^{(1-\varepsilon)N_u(x_j,r)} \max_{\mathbb{B}(x,r)} |u| \geq Mt^{(1-\varepsilon)N}.$$

By the covering lemma in Lemma 3.16,

$$\bigcup_{j=1}^{n+1} B(x_j, tr) \subset B(x_0, (1+\delta)tr),$$

in which $\delta = \mathrm{diam}(S)/(tr) = 1/(Kt) \to 0$ as $t \to \infty$. Therefore,

$$\max_{\mathbb{B}(x_j,tr)} |u| \leq \max_{\mathbb{B}(x_0,(1+\delta)tr)} |u| \quad \text{for all } j = 1, ..., n+1.$$

By Theorem 3.14 again,

$$\begin{aligned} \left(\frac{(1+\delta)t}{1+c_1}\right)^{(1+\varepsilon)N_u(x_0,(1+\delta)tr)} &\geq \frac{\max_{\mathbb{B}(x_0,(1+\delta)tr)} |u|}{\max_{\mathbb{B}(x_0,(1+c_1)r)} |u|} \\ &\geq \frac{\max_{\mathbb{B}(x_j,tr)} |u|}{\max_{\mathbb{B}(x_0,(1+c_1)r)} |u|} \\ &\geq \frac{Mt^{(1-\varepsilon)N}}{M} \\ &= t^{(1-\varepsilon)N}. \end{aligned} \tag{3.4}$$

Recall that a, K, c_1 are fixed. Now we choose the four parameters $t, \delta, \varepsilon, C$ in order.
(1). Choose $t > 2$ so that $\delta = 1/(Kt) < c_1/2$. Then

$$\frac{1+c_1/2}{1+c_1} = t^{-c_2},$$

in which

$$c_2 = \log_t\left(\frac{1+c_1}{1+c_1/2}\right) \in (0,1).$$

Hence,

$$\frac{(1+\delta)t}{1+c_1} \le \frac{t(1+c_1/2)}{1+c_1} = t^{1-c_2}.$$

Then (3.4) reduces to

$$N_u(x_0,(1+\delta)tr) \ge \frac{1-\varepsilon}{(1+\varepsilon)(1-c_2)}N.$$

(2). Choose $\varepsilon \in (0, c_2)$. Set $c = (c_2-\varepsilon)/2$. Hence,

$$\frac{1-\varepsilon}{(1+\varepsilon)(1-c_2)} \ge \frac{1-\varepsilon}{1-c_2} \ge 1+c,$$

which implies that

$$N_u(x_0,(1+\delta)tr) \ge \frac{1-\varepsilon}{(1+\varepsilon)(1-c_2)}N \ge (1+c)N.$$

(3). The lemma is done by choosing $C = (1+\delta)tK$.

□

$\boxed{\textbf{Problems}}$.

3-5. Let S be an equilateral triangle in $\mathbb{R}^2$ with side-length l. Let x_1, x_2, x_3 be the three vertices and x_0 be the barycenter. Find the relative width ω of S. Then prove that for all $K > 1$, there is positive constant $c_1 = c_1(K)$ such that

$$B(x_0,(1+c_1)r) \subset \bigcup_{j=1}^{3} B(x_j, r), \quad \text{in which } r = K \cdot \mathrm{diam}(S).$$

3-6. Prove Lemma 3.17. Try to prove it in $\mathbb{R}^2$ first.

3.7. The hyperplane lemma

Recall that the simplex lemma in Section 3.6 asserts that the doubling index at the vertices "add up" to the one at the barycenter for harmonic functions, if the simplex is non-degenerate. We also know, according to the example $(x+iy)^k$, the non-degeneracy condition can not be dropped.

From now on, we work with cubes instead of balls. We use Q and q to denote cubes. Define the doubling index of u at Q as

$$N_u(Q) = \sup_{x\in Q, r\in(0,\mathrm{diam}(Q))} N_u(x,r) = \sup_{x\in Q, r\in(0,\mathrm{diam}(Q))} \left\{\log_2\left(\frac{\max_{\mathbb{B}(x,2r)}|u|}{\max_{\mathbb{B}(x,r)}|u|}\right)\right\}. \tag{3.5}$$

If there is only one function in question, then we omit the subscript u.

Notice that to define $N_u(Q)$, we need u to be defined in a larger domain than Q, for example, $10nQ$ which is sufficiently large to contain $\mathbb{B}(x,2r)$ for all $x \in Q$ and $r < \mathrm{diam}(Q)$. Here, kQ means the cube with the same center as Q and side-length as k times the side-length of Q. However, this issue will not concern us, as we will always assume that u is defined in a sufficiently large domain.

Remark. In the view of the doubling index of balls, it is probably more natural to define the doubling index of cubes by

$$\log_2\left(\frac{\max_{2Q}|u|}{\max_Q|u|}\right).$$

However, such definition is not friendly for proving additivity (such as the simplex lemma in Section 3.6) or monotonicity.

Whereas the definition (3.5) of the doubling index of cubes has several advantages, including the immediate monotonicity property: If $q\subset Q$, then $N_u(q)\le N_u(Q)$.

Remark (Cubes vs. balls). We provide a quick comparison between cubes and balls when applying to the additivity properties of the doubling index. Notice that cubes can be partitioned into subcubes. In particular, let $A\in\mathbb{N}$. Then a cube Q divides into A^n subcubes of equal size. While balls can not be partitioned into small balls, they are nicely suited for the covering lemmas in Section 3.6, which can be used to establish a certain form of additivity in the simplex lemma.

In summary,

- balls work well with covering properties as in Section 3.6 and the additivity appears in the form of

  the doubling index of small balls "add up" to the one of a big ball,

  if the small balls are positioned at the vertices of a non-degenerate simplex.
- cubes work well with partition properties as in this and next sections and the additivity appears in the form of

  the doubling index of subcubes "add up" to the one of the cube,

  if the subcubes are positioned according to some geometric constraints.

In this section, we prove the hyperplane lemma. It asserts that if the doubling index on all the subcubes q that are "on" the same hyperplane is greater than N, then the doubling index on the big cube Q must "add up" to at least $2N$. Again, similarly as for the simplex lemma in the view of the example $(x+iy)^k$, the condition of the hyperplane can not be dropped.

LEMMA 3.19 (The hyperplane lemma). *Let $Q=[-R/2,R/2]^n\subset\mathbb{R}^n$. Divide Q into A^n subcubes of equal size. Denote $q_{j,0}$, $j=1,\dots,A^{n-1}$, the subcubes that intersect the hyperplane $\{x_n=0\}$. Then there are positive constants $A_0=A_0(n)$ and $N_0=N_0(n)$ such that if $A\ge A_0$ is odd and $N_u(q_{j,0})\ge N\ge N_0$ for all $q_{j,0}$, then*

$$N_u(Q)>2N$$

for all harmonic functions u.

Remark.

- Here, we require A to be odd so there is no ambiguity in defining the subcubes that intersect the hyperplane $\{x_n=0\}$, i.e., the ones whose centers are on $\{x_n=0\}$.
- The hyperplane lemma remains valid if $\{x_n=0\}$ is replaced by any other hyperplane that is parallel to one of the faces of the cube. This can be done by simply rotating the whole space and noticing that the harmonic functions are still harmonic after the rotation.
- The key idea to prove the hyperplane lemma is to apply the Cauchy uniqueness theorem, that is, if a harmonic function u and its derivative $\partial_\nu u$ (i.e., the Cauchy data) are both small on a piece of the hyperplane $\{x_n=0\}$, then u must be small in a neighborhood of $\{x_n=0\}$. In the application to prove the hyperplane lemma, the doubling index of the

subcubes that intersect $\{x_n = 0\}$ controls u and $\partial_\nu u$ on $\{x_n = 0\}$. It in turn determines the size of u in a neighborhood of $\{x_n = 0\}$, say Q/K for some large K. Then the doubling index of Q is controlled, as it can be estimated by the growth rate of $|u|$ from Q/K to Q.

- The hyperplane lemma follows from careful analysis of geometric objects such as cubes as well as balls. In addition, it needs the quantitative Cauchy uniqueness theorem 2.16: There exist constant $C > 0$ and $\varepsilon \in (0,1)$ such that if a harmonic function u in q satisfies
$$\begin{cases} |u(x)| \le 1, & \text{if } x \in q, \\ |u(x)| \le \varepsilon, & \text{if } x \in F, \\ |\partial_\nu u(x)| \le \frac{\varepsilon}{r}, & \text{if } x \in F, \end{cases}$$
for some $\varepsilon \ge 0$, then
$$\sup_{\frac{1}{2}q} |u| \le C\varepsilon^\alpha.$$
Here, F is a face of q, r is the side-length of q, and $\frac{1}{2}q$ is the middle cube of q with half side-length.

Proof of Lemma 3.19. Without loss of generality, we assume that $R = 1$. (In the general case, set $v(x) = u(x/R)$ so v is harmonic in $[-1/2, 1/2]^n$.) We divide the proof into four steps according to the plan mentioned above. Denote
$$M = \max_{\mathbb{B}(0,1/8)} |u|.$$

Step 1. Estimate $|u|$ on a piece of the hyperplane $\{x_n = 0\}$. Of course, $|u(x)| \le M$ on the piece $\mathbb{B}(0,1/8) \cap \{x_n = 0\}$. But this does not take into account of the doubling index near the hyperplane $\{x_n = 0\}$. To exploit the doubling index, we consider $\max_{2q_{j,0}} |u|$ and use it to control $|u(x)|$ for which $x \in q_{j,0}$ so there is doubling action involved.

Notice that $N_u(q_{j,0}) \ge N$, there is $x_j \in q_{j,0}$ and $r_j < \operatorname{diam}(q_j) = \sqrt{n}/A$ such that $N_u(x_j, r_j) \ge N \ge N_0$. Hence, elementary Euclidean geometry yields that
$$2q_{j,0} \subset \mathbb{B}\left(x_j, \frac{4\sqrt{n}}{A}\right).$$

Let $x \in \mathbb{B}(0,1/16) \cap \{x_n = 0\}$. Then there is $q_{j,0} \ni x$ such that $q_{j,0} \subset \mathbb{B}(0,1/12)$ (if $1/A < 1/12 - 1/16$, i.e., $A > 36$). Thus, $x_j \in \mathbb{B}(0,1/12)$ and $\mathbb{B}(x_j, 1/32) \subset \mathbb{B}(0,1/8)$ since $x_j \in q_{j,0} \subset \mathbb{B}(0,1/12)$. Therefore, $\max_{\mathbb{B}(x_j,1/32)} |u| \le M$.

We now require $4\sqrt{n}/A < 1/32$, i.e., $A \ge A_0(n) > 128\sqrt{n}$. Using the monotonicity formula of doubling index in Theorem 3.14 with $\varepsilon = 1/2$,
$$\frac{\max_{\mathbb{B}(x_j,1/32)} |u|}{\max_{\mathbb{B}\left(x_j, \frac{4\sqrt{n}}{A}\right)} |u|} \ge \left(\frac{A}{128\sqrt{n}}\right)^{\frac{N_u\left(x_j, \frac{4\sqrt{n}}{A}\right)}{2}} \ge \left(\frac{A}{128\sqrt{n}}\right)^{\frac{N_u(x_j,r_j)}{2}} \ge \left(\frac{A}{128\sqrt{n}}\right)^{\frac{N}{2}},$$
in which we also use the weak monotonicity of the doubling index in Theorem 3.13:
$$N_u\left(x_j, \frac{4\sqrt{n}}{A}\right) \ge N_u(r_j, r_j),$$
since $r_j < \operatorname{diam}(q_{j,0}) = \sqrt{n}/A$. We then have that
$$\max_{2q_{j,0}} |u| \quad \le \quad \max_{\mathbb{B}\left(x_j, \frac{4\sqrt{n}}{A}\right)} |u|$$

$$\leq \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot \max_{\mathbb{B}\left(x_j, \frac{1}{32}\right)} |u|$$

$$\leq \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M.$$

Therefore, $x \in \mathbb{B}(0, 1/16) \cap \{x_n = 0\}$,

$$|u(x)| \leq \max_{2q_{j,0}} |u| \leq \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M.$$

Step 2. Estimate $|\partial_\nu u|$ on $\mathbb{B}(0, 1/16) \cap \{x_n = 0\}$. To this end, we use the estimates of derivatives of harmonic functions in Theorem 2.8. That is,

$$|\partial_{x_j} u(x)| \leq \frac{C_1}{r^{n+1}} \int_{B(x,r)} |u| = \frac{4n}{\alpha_n r^{n+1}} \int_{B(x,r)} |u|.$$

For $x \in q_{j,0} \cap \{x_n = 0\}$, we choose $B(x, 1/A) \subset 2q_{j,0}$ in the above integral, where $1/A$ is the side-length of $q_{j,0}$. That is,

$$|\partial_{x_j} u(x)| \leq 4nA \cdot \max_{\mathbb{B}(x,1/A)} |u| \leq 4nA \cdot \max_{2q_{j,0}} |u|.$$

Using the estimate of $\max_{2q_{j,0}} |u|$ in Step 1, we have that

$$|\partial_{x_j} u(x)| \leq 4nA \cdot \max_{2q_{j,0}} |u| \leq 4nA \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M$$

for all $j = 1, ..., n$. Therefore, for all $x \in B(0, 1/16) \cap \{x_n = 0\}$,

$$|\nabla u(x)| \leq \sum_{j=1}^{n} |\partial_{x_j} u(x)| \leq 4n^2 A \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M,$$

which of course also bounds $|\partial_\nu u(x)|$.

Step 3. Application of the quantitative Cauchy uniqueness theorem. From Steps 1 and 2, we know that on $\mathbb{B}(0, 1/16) \cap \{x_n = 0\}$,

$$|u| \leq \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M \quad \text{and} \quad |\partial_\nu u| \leq 4n^2 A \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M.$$

We see that

$$4n^2 A \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} \cdot M \leq M \cdot 2^{-c_1(n) N \log A},$$

in which $c_1(n)$ is a positive constant depending only on n.

Now we set up a cube q to apply the quantitative Cauchy uniqueness theorem. Let q be a cube with side-length $1/(16\sqrt{n})$ in the upper half space $\{x_n \geq 0\}$ such that $q \subset \mathbb{B}(0, 1/16)$ and

$$\mathbb{B}\left(0, \frac{1}{32\sqrt{n}}\right) \cap \{x_n = 0\} \subset \partial q \cap \{x_n = 0\} \subset \mathbb{B}\left(0, \frac{1}{16}\right) \cap \{x_n = 0\}.$$

That is, q has a face F on the hyperplane $\{x_n = 0\}$

Let $v = u/M$. Then $|v| \le 1$ in q. Moreover, on the face $F \subset \mathbb{B}(0, 1/16) \cap \{x_n = 0\}$, notice that the side-length of q is $1/(16\sqrt{n})$,

$$|v| \le \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} = 2^{-c(n)N\log A} := \varepsilon,$$

and

$$|\partial_\nu u| \le 4n^2 A \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}} = \frac{\frac{n^2 A}{4\sqrt{n}} \cdot \left(\frac{128\sqrt{n}}{A}\right)^{\frac{N}{2}}}{\frac{1}{16\sqrt{n}}} \le \frac{\varepsilon}{\frac{1}{16\sqrt{n}}},$$

in which $c_1(n)$ is a positive constant depending only on n.

Using the quantitative Cauchy uniqueness theorem, $|v| \le \varepsilon^\alpha$ in $q/2$. So

$$\max_{\frac{1}{2}q} |u| \le M \cdot \varepsilon^\alpha = M \cdot 2^{-c_1(n)\alpha N \log A}.$$

Denote p the center of the cube q. Then since $q/2$ has side-length $1/(32\sqrt{n})$,

$$\mathbb{B}\left(p, \frac{1}{64\sqrt{n}}\right) \subset \frac{1}{2}q.$$

So

$$\max_{\mathbb{B}\left(p, \frac{1}{64\sqrt{n}}\right)} |u| \le M \cdot 2^{-c_1(n)\alpha N \log A}.$$

Step 4. Estimate $N_u(p, 1/2)$ and $N_u(Q)$. Recall that p is the center of q that satisfies $p \in \mathbb{B}(0, 1/32)$. Then $\mathbb{B}(p, 1/2) \supset \mathbb{B}(0, 1/8)$ so $\max_{\mathbb{B}(p,1/2)} \ge M = \max_{\mathbb{B}(0,1/8)}$. Therefore,

$$\frac{\max_{\mathbb{B}\left(p,\frac{1}{2}\right)} |u|}{\max_{\mathbb{B}\left(p,\frac{1}{64\sqrt{n}}\right)} |u|} \ge 2^{c_1(n)\alpha N \log A}.$$

But according to the monotonicity formula of the doubling index in Theorem 3.14 with $\varepsilon = 1/2$,

$$\frac{\max_{\mathbb{B}\left(p,\frac{1}{2}\right)} |u|}{\max_{\mathbb{B}\left(p,\frac{1}{64\sqrt{n}}\right)} |u|} \le (32\sqrt{n})^{\frac{3}{2}N_u(p,1/2)} = 2^{c_2(n)N_u(p,1/2)},$$

in which $c_2(n)$ is a positive constant depending only on n. Putting the above two inequalities together:

$$c_2(n)N_u(p, 1/2) \ge c_1(n)\alpha N \log A,$$

which implies that

$$N_u(p, 1/2) > 2N,$$

if $A \ge A_0(n)$ is chosen large enough. The theorem is then completed by noticing that

$$N_u(Q) = \sup_{x \in Q, r \in (0, \mathrm{diam}(Q))} N_u(x, r) \ge N(p, 1/2).$$

$\square$

By inductively applying the hyperplane lemma, we prove the following corollary.

COROLLARY 3.20. *Let $Q = [-R/2, R/2]^n \subset \mathbb{R}^n$ and $\varepsilon > 0$. Divide Q into A^n subcubes of equal size. Denote $q_{j,0}$, $j = 1, ..., A^{n-1}$, the subcubes that intersect the hyperplane $\{x_n = 0\}$. Then there are positive constants $A_0 = A_0(\varepsilon, n)$ and $N_0 = N_0(n)$ such that if $A \ge A_0$ is odd and $N_u(Q) = N \ge N_0$, then*

$$\#\{j : N_u(q_{j,0}) \ge N/2\} < \varepsilon A^{n-1}$$

for all harmonic functions u.

That is, among all the subcubes that intersect $\{x_n = 0\}$, almost all subcubes (i.e., all of them except ε proportion) have doubling index $< N/2$.

Proof. We prove the corollary inductively.

Step 1. Divide Q into A_0^n subcubes of equal size. Denote $q_{j,0}$ the subcubes that intersect $\{x_n = 0\}$. By the hyperplane lemma 3.19, there are constant $A_0 = A_0(n)$ and $N_0 = N_0(n)$ such that if $N_u(Q) = N \geq N_0$, then there is at least one subcube $q_{j,0}$ with doubling index $< N/2$. (If not, then all of $q_{j,0}$ have doubling index $\geq N/2$ so $N_u(Q) > N$ which is not possible.)

Denote M_1 the number of A_0^{n-1} subcubes that intersect $\{x_n = 0\}$ and with doubling index $\geq N/2$. We then have that

$$M_1 \leq A_0^{n-1} - 1.$$

Step 2. Divide Q into A_0^{2n} subcubes, i.e., each subcube in Step 1 is divided into A_0^n subcubes of equal size. Notice that if a cube has doubling index $< N/2$, then all of its subcubes have doubling index $< N/2$. Equivalently, the subcubes with doubling index $\geq N/2$ can only come from partitions of cubes with doubling index $\geq N/2$.

Denote M_2 the number of subcubes that intersect $\{x_n = 0\}$ and with doubling index $\geq N/2$. We then have that

$$M_2 \leq M_1 \cdot (A_0^{n-1} - 1).$$

Step k. Divide Q into A_0^{kn} subcubes, i.e., each subcube in Step $k-1$ is divided into A_0^n subcubes of equal size. Denote M_k the number of subcubes that intersect $\{x_n = 0\}$ and with doubling index $\geq N/2$. We then have that

$$\begin{aligned} M_k &\leq M_{k-1} \cdot (A_0^{n-1} - 1) \\ &\leq (A_0^{n-1} - 1)^k \\ &\leq A_0^{k(n-1)} \left(1 - \frac{1}{A_0^{n-1}}\right)^k \\ &\leq \varepsilon \cdot \left(A_0^k\right)^{n-1}, \end{aligned}$$

if $k = k(\varepsilon, A_0)$ is large enough that

$$\left(1 - \frac{1}{A_0^{n-1}}\right)^k < \varepsilon.$$

Now set $A_0(\varepsilon, n) = A_0^k$. Then in Step k, $M_k \leq \varepsilon A_0^{n-1}$ and we are done. $\square$

Remark. The choice of the hyperplane $\{x_n = 0\}$ in the hyperplane lemmas are really arbitrary. Let Q be a cube and $P \subset Q$ be a hyperplane that is parallel to one of its faces. If $N_u(Q) = N \geq N_0$, then the number of subcubes that intersect P and have doubling index $\geq N/2$ is always $< \varepsilon A^{n-1}$.

In particular, one can find A parallel hyperplanes to $\{x_n = 0\}$ such that any subcube intersects one and only one of them. On each hyperplane, there are $< \varepsilon A^{n-1}$ subcubes with doubling index $\geq N/2$. It is follows that the total number of subcubes with doubling index $\geq N/2$ is $< \varepsilon A^{n-1} \cdot A = \varepsilon A^n$, i.e., only ε proportion of the A^n subcubes.

In the next section, we combine the hyperplane lemma and the simplex lemma to prove a stronger (in certain sense) result in counting the number of subcubes with large doubling index (say, larger than $N/(1 + c)$ for some $c > 0$.)

3.8. Weak additivity of the doubling index

In this section, we prove the following crucial theorem, which is directly responsible for the nodal size estimate of harmonic functions. (See the next chapter.)

THEOREM 3.21. *There are constants $c = c(n) > 0$, $N_0 = N_0(n) > 0$, and an integer $A = A(n) > 0$ such that if we divide any cube Q into A^n subcubes q of equal size, then*

$$\#\left\{q : N_u(q) > \frac{N_u(Q)}{1+c}\right\} < \frac{1}{2}A^{n-1}$$

for all harmonic functions u with $N_u(Q) \geq N_0$.

Remark. The above theorem asserts that the number of subcubes with large doubling index (say, larger than $N_u(Q)/(1+c)$) is smaller than $A^{n-1}/2$. We again call this theorem a weak version of additivity of the doubling index. In fact, this additivity property takes an opposite direction of "adding up", that is, the doubling index of a large portion (i.e., except at most $A^{n-1}/2$) of subcubes "drop from" the one of the cube.

The proof of Theorem 3.21 is intuitively straightforward but is (very) complicated in details. We present the intuition and the attempts before the actual proof to clarify the main idea.

Intuition. The proof of Theorem 3.21 is an ingenious combination of the simplex lemma 3.18 and the hyperplane lemma 3.19 (or rather, its corollary 3.20). Both of these two lemmas provide constraints about where the balls/cubes with large doubling index can be. We need their full force to establish Theorem 3.21. We begin from the intuition. For the sake of simplicity, we assume the simplex lemma (which is stated for balls) for cubes.

- Intuition I: The simplex lemma states that if there is a non-degenerate (i.e., non-flat) simplex and there are cubes with doubling index $\geq N/(1+c)$ at the vertices, then the doubling index at the barycenter is $> N$. This can not happen if the doubling index of the (largest) cube Q is N. As a consequence, the subcubes with doubling index $\geq N/(1+c)$ have to be squeezed to a thin neighborhood of some hyperplane.
- Intuition II. The hyperplane lemma (Corollary 3.20) states that the number of subcubes that intersect a hyperplane and have doubling index $\geq N/2$ is $< \varepsilon A^{n-1}$. As a consequence, the subcubes that are in a thin neighborhood of a hyperplane and have doubling index $\geq N/(1+c) \geq N/2$ can be controlled by $A^{n-1}/2$. (Let us assume $0 < c < 1/10$.)

We remark that the coefficient $1/2$ of $A^{n-1}/2$ in Theorem 3.21 is rather arbitrary. The importance is that $1/2 < 1$ and the order is A^{n-1}, i.e., the total number of subcubes with large doubling index is less than a fraction of the ones that intersect a fixed hyperplane.

Attempts. Let us get more serious and ask more detailed questions about realization of the intuition. We recall the simplex lemma:

Remark (The simplex lemma). Let $a > 0$ and $K = K(a,n) \geq 2/a$. Suppose that $\omega(S) > a$ and $r = K \cdot \operatorname{diam}(S)$. Then there are positive constants $c_0 = c_0(a,n)$, $C_0 = C_0(a,n) \geq K$, and $N_0 = N_0(a,n)$ such that if for each $j = 1, ..., n+1$, $N_u(x_j, r_j) \geq N \geq N_0$ for some $0 < r_j \leq r/2$, then

$$N_u\left(x_0, C_0 \cdot \operatorname{diam}(S)\right) > (1+c_0)N$$

for all harmonic functions u.

- In Intuition I, to squeeze subcubes to a hyperplane, we need to use simplexes that are rather degenerate (i.e., nearly flat). However, the simplex lemma 3.18 requires the simplex to be non-degenerate. We need to balance these issues in a quantitative manner.
- In Intuition II, how thin must the neighborhood of the hyperplane be? According to Corollary 3.20, each hyperplane "contains" εA^{n-1} subcubes with large doubling index and the hyperplanes separate by distance A^{-1}. Therefore, a neighborhood no thicker than $1/(2\varepsilon A)$ will do. That is, our choices of the parameters ε and A need to be related in a quantitative manner.

There is another issue about the application of the simplex lemma. Namely, if a simplex S is non-degenerate, then there is a ball centered at the barycenter with strictly larger doubling index than the ones centered at the vertices. However, geometrically we have the pay higher prices if the simplex is more degenerate (i.e., $a \to 0$). More precisely, we need $K \to \infty$ (i.e., balls centered at the barycenter and at the vertices all have to blow up).

To apply the simplex lemma as in Intuition I, we eventually need to squeeze cubes/balls in Q with large doubling index to a thin neighborhood of some hyperplane, which demands very degenerate simplexes. But if we begin from a very degenerate simplex S in Q, then the balls in the simplex lemma with radius $K \cdot \operatorname{diam}(S)$ as $K \to \infty$ may go beyond Q. To remedy this issue, we first partition Q into subcubes q small enough so that blown-up balls involving a very degenerate S in q are still contained in Q.

Proof. With the direction of the above intuition and attempts, we prove Theorem 3.21 in four stages.

3.8.1. Stage 1: Setup.

In this Stage, we set up the three parameters in Theorem 3.21.

- Let $c > 0$, whose value will be chosen in Stage 2. We assume $c < 1/10$ and practically it will be small.
- Let $N_0 > 0$, whose value will be chosen in Stages 2 and 3.
- Let $A > 0$ be an integer, whose value will be chosen in Stage 4.

Ultimately, we choose c, N_0, and A as parameters that depend only on the dimension n. To this end, we need to introduce some intermediate parameters:

- In Stage 1, $A_0 \geq 3$ is an odd integer, which denotes the partition scale, i.e., a cube is partitioned into A_0^n subcubes of equal size.
- In Stage 2, $j > 0$ is an integer, which denotes the partition step, i.e., a cube in j-th step is partitioned into A_0^n subcubes of equal size in $(j+1)$-th step. The subcubes in j-th step gets smaller as j gets larger.
- In Stage 2, $\omega_0 > 0$, which controls the degeneracy of the simplexes.
- In Stage 3, $\varepsilon > 0$, which controls the thickness of the neighborhood of some hyperplane.

In Stage 4, these intermediate parameters will be chosen to depend only on n to achieve our ultimate goal.

We also set up the partition terminology. Let $A_0 \geq 3$ be an odd integer. We inductively partition Q into subcubes of equal size. That is, in j-th step, Q is partitioned into A_0^{nj} subcubes of equal size:

$$\{Q_{i_1,\dots,i_j} : i_1, \dots, i_j = 1, \dots, A_0^n\},$$

in which $\operatorname{diam}(Q_{i_1,\dots,i_j}) = \operatorname{diam}(Q)/A_0^j$.

Definition (Good cubes and bad cubes). We say a subcube $q = Q_{i_1,\dots,i_j}$ in j-th step is good if $N(q) \leq N(Q)/(1+c)$ and is bad otherwise.

According to the definition of the doubling index, if a cube in j-th step is good, then all of its subcubes in $(j+1)$-th step are good.

3.8.2. Stage 2: Application of the simplex lemma. In this stage, we fix a cube $q = Q_{i_1,...,i_j}$ in j-th step and denote q_i, $i = 1, ..., A_0^n$ its subcubes in $(j+1)$-th step. We use the simplex lemma to prove that the bad cubes q_i in q are contained an arbitrarily thin neighborhood of some hyperplane. As mentioned in the Attempts, we need j large so q is small and it is safe to apply the simplex lemma for $q \subset Q$.

Denote

$$F := \left\{ x \in q : N(x, r) > \frac{N(Q)}{1+c} \quad \text{for some } 0 < r \leq \operatorname{diam}(q_i) = \frac{\operatorname{diam}(q)}{A_0} \right\} \subset q.$$

Define the relative width of F (in q) as

$$\widetilde{\omega}(F) = \frac{\operatorname{width}(F)}{\operatorname{diam}(q)},$$

in which $\operatorname{width}(F)$ is the minimal distance between a pair of parallel hyperplanes such that F is contained between them.

If $\widetilde{\omega}(F) > 0$, then one can find a simplex with vertices in F that is non-degenerate. Recall that $\omega(S) = \operatorname{width}(S)/\operatorname{diam}(S)$ is the relative width of a simplex S.

PROPOSITION 3.22. *Let $F \subset q$ with $\widetilde{\omega}(F) > 0$. Then there is a simplex with vertices $x_1, ..., x_{n+1}$ in F such that*

$$\omega(S) > a \quad \textit{and} \quad \operatorname{diam}(S) > a \cdot \operatorname{diam}(q),$$

in which $a = a\left(\widetilde{\omega}(F), n\right) > 0$.

PROOF. See Problem 3-7. □

Combining Proposition 3.22 and the simplex lemma 3.18, we prove the following lemma.

LEMMA 3.23. *Let $\omega_0 > 0$. Then there are positive constants $j_0 = j_0(\omega_0, n)$ and $c_0 = c_0(\omega_0, n)$ such that if $j \geq j_0$ is an integer and $0 < c \leq c_0$, then $\widetilde{\omega}(F) < \omega_0$.*

In particular, this lemma asserts that F in q has to be squeezed to an arbitrarily thin (by ω_0) neighborhood of some hyperplane. Here, we need j large enough so $\operatorname{diam}(q) = \operatorname{diam}(Q)/A_0^j$ is small enough and we are safe to apply the simplex lemma in q.

PROOF OF LEMMA 3.23. We prove by contradiction. Assume that $\widetilde{\omega}(F) \geq \omega_0$. Then by Proposition 3.22, there is a simplex S with vertices $x_1, ..., x_{n+1}$ in F such that

$$\omega(S) > a \quad \text{and} \quad \operatorname{diam}(S) > a \cdot \operatorname{diam}(q),$$

in which $a = a\left(\omega_0, n\right) > 0$. Since $x_j \in F$,

$$N(x_j, r_j) > \frac{N(Q)}{1+c} \quad \text{for some } r_j \leq \operatorname{diam}(q).$$

We restate the simplex lemma here:

Remark (Simplex lemma). Let $a = a(\omega_0, n) > 0$ and $K = K(\omega_0, n) \geq 2/a$. Suppose that $\omega(S) > a$ and $r = K \cdot \operatorname{diam}(S)$. Then there are positive constants $c_1 = c_1(\omega_0, n)$, $C_1 = C_1(\omega_0, n) \geq K$, and $N_1 = N_1(\omega_0, n)$ such that if for each $j = 1, ..., n+1$, $N_u(x_j, r_j) \geq N \geq N_1$ for some $0 < r_j \leq r/2$, then

$$N_u\left(x_0, C_1 \cdot \operatorname{diam}(S)\right) > (1 + c_1)N$$

for all harmonic functions u.

We next choose the parameters N_0 in Theorem 3.21 and j_0, c_0 in Lemma 3.23 according to the ones in the simplex lemma, in order to arrive at a contradiction.

(1). Choose $N_0 = N_0(\omega_0, n) = 2N_1(\omega_0, n)$ in Theorem 3.21. Since $c < 1/10$, $N_u(Q)/(1+c) \geq N_u(Q)/2 \geq N_0/2 = N_1$. Hence,

$$N(x_j, r_j) > \frac{N(Q)}{1+c} \geq N_1.$$

Moreover, since $K \geq 2/a$ in the simplex lemma, we have that

$$r_j \leq \operatorname{diam}(q) < \frac{1}{a} \cdot \operatorname{diam}(S) \leq \frac{K}{2} \cdot \operatorname{diam}(S).$$

Applying the simplex lemma,

$$N(x_0, C_1 \cdot \operatorname{diam}(S)) \geq \frac{(1+c_1)N}{1+c}.$$

(2). Choose $c_0 = c_0(\omega_0, n) = c_1(\omega_0, n)/2$. Hence, if $c \leq c_0$, then

$$N(x_0, C_1 \cdot \operatorname{diam}(S)) > \frac{(1+c_1)N}{1+c} > N.$$

(3). Choose $j_0 = j_0(\omega_0, n)$ such that $3^{j_0} > C_1 = C_1(\omega_0, n)$. Thus, since $A_0 \geq 3$,

$$C_1 \cdot \operatorname{diam}(S) \leq C_1 \cdot \operatorname{diam}(q) = C_1 \cdot \frac{\operatorname{diam}(Q)}{A_0^j} < C_1 \cdot \frac{\operatorname{diam}(Q)}{3^j} \leq \operatorname{diam}(Q),$$

if $j \geq j_0$. But then $N(x_0, C_1 \cdot \operatorname{diam}(S)) > N$ for $x_0 \in q \subset Q$ and $C_1 \cdot \operatorname{diam}(S) < \operatorname{diam}(Q)$ implies that $N(Q) > N$, which is not possible. The lemma is complete.

(3'). For future application in Stage 3, we will choose an absolute constant j_0' such that if $j \geq j_0'$, then

$$N(100q) \leq \frac{20}{19} \cdot N(Q) \quad \text{for each } q \subset Q. \tag{3.6}$$

By Theorem 3.15 with $\delta = 1/20$, there is $K \geq 1$ and $N_2 = N_2(n)$ such that

$$N(y, 2Kr) \geq \frac{20}{19} \cdot N(x, r),$$

if $N(x, r) \geq N_2$. We modify our choice of N_0 in (1) above so $N_0 = N_0(\omega_0, n) \geq N_2$. This is of course harmless since N_2 depends only on n.

If $N(x, r) < N_0$ for all $x \in 100q$ and $r < 100\operatorname{diam}(q) = 100\operatorname{diam}(Q)/A_0^j := r$, then $N(100q) \leq N_0 \leq N(Q)$ and we are done.

If $N(x, r) \geq N_0 \geq N_2$ for some $x \in 100q$. Since $q \subset Q$, there is $y \in Q$ and $|x - y| \leq 100\operatorname{diam}(q)$. Hence,

$$N(y, 2Kr) \geq \frac{20}{19} \cdot N(x, r),$$

in which

$$2Kr = \frac{200K \cdot \operatorname{diam}(Q)}{A_0^j} \leq \frac{200K \cdot \operatorname{diam}(Q)}{3^j} \leq \operatorname{diam}(Q),$$

if we choose $j > j_0'$ and $3^{j_0'} > 2K$. Now $N(Q) \geq N(y, 2Kr)$ and (3.6) is proved.

Combining (3) and (3'), we choose $j_0 = j_0(\omega_0, n) \geq j_0'$ so (3.6) is valid.

$\square$

3.8.3. Stage 3: Application of the hyperplane lemma. In this stage, we use the hyperplane lemma to prove that the number of bad cubes q_i in q,

$$\#\left\{q_i : N(q_i) > \frac{N(Q)}{1+c}\right\} < \frac{1}{2}A_0^{n-1}.$$

The whole stage is a proof by contradiction.

Notice that in Stage 2, we only assume that A_0 is an odd integer and is at least 3. Now let $\omega_0 > 0$ such that $A_0 = 1/\omega_0$ is an odd integer. Recall our notation that q is partitioned into A_0^n subcubes q_i of equal size.

By Lemma 3.23, $\widetilde{\omega}(F) = \mathrm{width}(F)/\mathrm{diam}(q) < \omega_0$. Hence, F is contained in a $\omega_0 \cdot \mathrm{diam}(q)$ neighborhood of a hyperplane $P \subset q$. If q_i is bad, then

$$N(q_i) = \sup_{x\in q_i, r\in(0,\mathrm{diam}(q_i))} N(x,r) > \frac{N(Q)}{1+c},$$

which implies that $N(x,r) > N(Q)/(1+c)$ for some $x \in q_i$ and $r \leq \mathrm{diam}(q_i)$, i.e., $x \in F$. We then conclude that the bad cubes are all near the hyperplane P, since F is contained in a neighborhood of P. More precisely, for any point y in a bad subcube, we have that

$$\begin{aligned}\mathrm{dist}(y,P) &\leq |y-x| + \mathrm{dist}(x,P)\\ &\leq \mathrm{diam}(q_i) + \mathrm{dist}(F,P)\\ &\leq \mathrm{diam}(q_i) + \omega_0 \cdot \mathrm{diam}(q)\\ &= 2\mathrm{diam}(q_i),\end{aligned}$$

since $\mathrm{diam}(q_i) = \mathrm{diam}(q)/A_0$ and $\omega_0 = 1/A_0$. That is, all bad subcubes in q are contained in the $2\mathrm{diam}(q_i)$ neighborhood of the hyperplane P. However, the hyperplane P is not necessarily parallel to one of the faces of q. So we can not apply the hyperplane lemma yet. This minor issue can be overcome by a geometric modification:

Let $\widetilde{q} \supset q$ such that the hyperplane P is parallel to one face of $\widetilde{q}$ and contains the center of $\widetilde{q}$. This can be done by letting $\mathrm{diam}(\widetilde{q}) = 10\mathrm{diam}(q)$. Then we see that $q \subset \widetilde{q} \subset 100q$. Now partition $\widetilde{q}$ into A_0^n subcubes $\widetilde{q}_i$ of equal size. Denote $\widetilde{q}_{i,0}$ the subcubes that intersect P.

We know that each bad subcube q_i is in the $2\mathrm{diam}(q_i)$ neighborhood of P. Whereas $\widetilde{q}_{i,0}$ have centers on P and $\mathrm{diam}(\widetilde{q}_{i,0}) = 10\mathrm{diam}(q_i)$. This means that each bad subcube q_i intersect at most $C = C(n)$ of $\widetilde{q}_{i,0}$ and is covered by them. Furthermore, if q_i is bad and intersect $\widetilde{q}_{i,0}$, then $\widetilde{q}_{i,0}$ is also bad. Namely,

$$\#\left\{q_i : N(q_i) > \frac{N(Q)}{1+c}\right\} \leq C(n)\cdot\#\left\{\widetilde{q}_{i,0} : N(\widetilde{q}_{i,0}) > \frac{N(Q)}{1+c}\right\}.$$

Assume that

$$\#\left\{q_i : N(q_i) > \frac{N(Q)}{1+c}\right\} \geq \frac{1}{2}A_0^{n-1}.$$

It then follows that

$$\#\left\{\widetilde{q}_{i,0} : N(\widetilde{q}_{i,0}) > \frac{N(Q)}{1+c}\right\} \geq \frac{1}{2C(n)}\cdot A_0^{n-1} \geq \varepsilon A_0^{n-1}, \tag{3.7}$$

if we choose $\varepsilon = \varepsilon(n) = 1/(4C(n))$. This seems to induce contradiction with the hyperplane lemma so we restate the hyperplane lemma in Corollary 3.20:

Remark (Hyperplane lemma). Let $\varepsilon > 0$. Then there are odd integer $A_0 = A_0(\varepsilon, n)$ and $N_2 = N_2(n) > 0$ such that if $N(\widetilde{q}) \geq N_2$, then

$$\#\{\widetilde{q}_{i,0} : N(\widetilde{q}_{j,0}) \geq N(\widetilde{q})/2\} < \varepsilon A_0^{n-1}.$$

In order to apply the hyperplane lemma, we modify our choice of N_0 in Theorem 3.21 so that $N_0 \geq 2N_2$ in the hyperplane lemma. This is harmless since $N_2 = N_2(n)$. This means that $N(Q) \geq N_0 \geq 2N_2$. Because there is at least one bad subcube q_i in $q \subset \widetilde{q}$ (otherwise we are done), we have that

$$N(\widetilde{q}) \geq N(q) \geq N(q_i) \geq \frac{N(Q)}{1+c} \geq \frac{2N_2}{1+c} \geq N_2.$$

since $c < 1/10$. Therefore, we can apply the hyperplane lemma:

$$\#\{\widetilde{q}_{i,0} : N(\widetilde{q}_{j,0}) \geq N(\widetilde{q})/2\} < \varepsilon A_0^{n-1}.$$

Comparing with (3.7), we then conclude that

$$\frac{N(\widetilde{q})}{2} \geq \frac{N(Q)}{1+c},$$

which is

$$N(\widetilde{q}) \geq \frac{2N(Q)}{1+c} \geq \frac{20}{11} \cdot N(Q).$$

However, since $q \subset \widetilde{q} \subset 100q$,

$$N(\widetilde{q}) \leq N(100q) \leq \frac{20}{19} \cdot N(Q),$$

as a consequence of (3.6). We finally arrive at a contradiction.

3.8.4. Stage 4: Completion. In this stage, we complete the proof of Theorem 3.21. We use the following diagram to demonstrate the dependence relations among the parameters.

$$\boxed{\begin{tikzcd}
 & & & N_0(\omega_0) \arrow[d] & \\
n & \varepsilon(n) \arrow[l] & A_0(\varepsilon) \arrow[l] & \omega_0(A_0) \arrow[l] & j_0(w_0) \arrow[l] \\
 & & & c(\omega_0) \arrow[u] & A(A_0, j_0) \arrow[u]
\end{tikzcd}}$$

That is, let $\varepsilon = \varepsilon(n)$. Then $A_0 = A_0(\varepsilon, n) = A_0(n)$, $\omega_0 = 1/A_0 = \omega_0(n)$, $j_0 = j_0(\omega_0, n) = j_0(n)$, $c = c(\omega_0, n) = c(n)$, and $N_0 = N_0(\omega_0, n)$ are chosen according to the rules specified in Stages 2 and 3. That is, we inductively partition Q into subcubes of equal size. Let $j \geq j_0$. Denote q a subcube in j-th step and q_i the subcubes of q in $(j+1)$-th step. Then we have that

$$\#\left\{q_i : N(q_i) > \frac{N(Q)}{1+c}\right\} \geq \frac{1}{2}A_0^{n-1}.$$

if $N(Q) \geq N_0$. For $j \geq j_0$, write M_j the number of subcubes in j-th step with doubling index $> N(Q)/(1+c)$. Compute that

$$\begin{aligned}
M_j &\leq \frac{1}{2}A_0^{n-1} \cdot M_{j-1} \\
&\leq \left(\frac{1}{2}A_0^{n-1}\right)^{j-j_0} \cdot M_{j_0} \\
&= \frac{M_{j_0}}{2^{(n-1)(j-j_0)}} \cdot \left(A_0^{j-j_0}\right)^{n-1}.
\end{aligned}$$

Set $A = A_0^{j-j_0}$. Then

$$M_j \leq \frac{M_{j_0}}{2^{(n-1)(j-j_0)}} \cdot A^{n-1} \leq \frac{1}{2}A^{n-1}$$

by choosing $j = j(A_0, j_0)$ large enough. Now $A = A_0^{j-j_0} = A(A_0, j_0) = A(n)$ since A_0 and j_0 depend only on n. Theorem 3.21 is complete.

$\boxed{\textbf{Problems}}$.

3-7. Prove Proposition 3.22.

CHAPTER 4

Upper bounds of the nodal size

In Chapter 3, we study the doubling index and its properties, in particular, we establish the crucial weak additivity formula of the doubling index in Theorem 3.21:

There are positive constants $c = c(n)$, $A = A(n)$ (an integer), and $N_0 = N_0(n)$ such that if $N_u(Q) \geq N_0$ and we divide any cube Q into A^n subcubes q of equal size, then

$$\# \left\{ q : N_u(q) > \frac{N_u(Q)}{1+c} \right\} < \frac{1}{2} A^{n-1}$$

for all harmonic functions u.

In this chapter, we apply this weak additivity formula of the doubling index to prove the upper bound of the nodal sizes of harmonic functions. As a consequence, the upper bound of the nodal size of eigenfunctions also follows.

4.1. Nodal size of harmonic functions

THEOREM 4.1 (Upper bound of the nodal size of harmonic functions). *There exist positive constants $\alpha = \alpha(n)$ and $C = C(n)$ such that in any cube $Q \subset \mathbb{R}^n$*

$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q) \leq C \cdot \operatorname{diam}(Q)^{n-1} N_u(Q)^\alpha$$

for all harmonic functions u.

Remark. The proof uses the subadditivity of the Hausdorff measure $\mathcal{H}^n$. That is, if $E = E_1 \cup E_2$, then

$$\mathcal{H}^n(E) \leq \mathcal{H}^n(E_1) + \mathcal{H}^n(E_2).$$

It is rather curious that no more properties of the Hausdorff measure are needed in the proof.

PROOF. We first remark that if $\mathcal{N}(u) \cap Q \neq \emptyset$, then there is $x \in Q$ such that $u(x) = 0$. Then the vanishing order $V_u(x) \geq 0$. Hence, $N_u(x, r) \to d + 1 \geq 1$ as $r \to 0$. It therefore follows that

$$N_u(Q) = \sup_{x \in Q, r \in (0, \operatorname{diam}(Q))} N_u(x, r) \geq 1.$$

Define

$$F(N) := \sup_{Q \subset \mathbb{R}^n \text{ and } N_u(Q) \leq N} \frac{\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q)}{\operatorname{diam}(Q)^{n-1}},$$

that is, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and all harmonic functions u such that $N_u(Q) \leq N$. We can assume that $N \geq 1$.

Due to Hardt-Simon [**HS**], we know that $F(N) \leq C_1 e^N < \infty$. The theorem is proved if we show that $F(N) \leq N^\alpha$ for some $\alpha = \alpha(n) > 0$.

Let A, c, and N_0 be the same in Theorem 3.21. We next show that

$$F(N) \leq 4A \cdot F\left(\frac{N}{1+c}\right)$$

for all $N \geq N_0$. Suppose the contrary, i.e., there is $N \geq N_0$ such that

$$F(N) > 4A \cdot F\left(\frac{N}{1+c}\right).$$

By the definition of $F(N)$, there are a cube Q and a harmonic function u in $2Q$ such that $N_u(Q) \leq N$ such that

$$\frac{\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q)}{\operatorname{diam}(Q)^{n-1}} > \frac{3}{4}F(N). \tag{4.1}$$

Partition Q into A^n subcubes Q_j, $j = 1, ..., A^n$, of equal size. Then $\operatorname{diam}(Q_j) = \operatorname{diam}(Q)/A$. Divide these subcubes into two groups:

- $\mathcal{G}_1 = \{Q_j : N/(1+c) < N_u(Q_j) \leq N\}$: For each $Q_j \in \mathcal{G}_1$, we have that
$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q_j) \leq F(N) \cdot \operatorname{diam}(Q_j)^{n-1} = F(N) \cdot \frac{\operatorname{diam}(Q)^{n-1}}{A^{n-1}}.$$
- $\mathcal{G}_2 = \{Q_j : N_u(Q_j) \leq N/(1+c)\}$: For each $Q_j \in \mathcal{G}_2$, we have that
$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q_j) \leq F\left(\frac{N}{1+c}\right) \cdot \operatorname{diam}(Q_j)^{n-1} = \frac{F(N)}{4A} \cdot \frac{\operatorname{diam}(Q)^{n-1}}{A^{n-1}},$$
in which we use
$$F\left(\frac{N}{1+c}\right) < \frac{F(N)}{4A},$$
by the assumption.

Furthermore, we know that $|\mathcal{G}_1| + |\mathcal{G}_2| = A^n$ so $|\mathcal{G}_2| \leq A^n$. By Theorem 3.21,

$$|\mathcal{G}_1| \leq \frac{1}{2}A^{n-1}.$$

Here, $|\mathcal{G}_1|$ and $|\mathcal{G}_2|$ stand for their sizes, i.e., their number of elements. Therefore, by the subadditivity of the Hausdorff measure, we have that

$$\begin{aligned}
&\mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q) \\
\leq\ & \sum_{Q_j \in \mathcal{G}_1} \mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q_j) + \sum_{Q_j \in \mathcal{G}_2} \mathcal{H}^{n-1}(\mathcal{N}(u) \cap Q_j) \\
\leq\ & |\mathcal{G}_1| \cdot \frac{F(N) \cdot \operatorname{diam}(Q)^{n-1}}{A^{n-1}} + |\mathcal{G}_2| \cdot \frac{F(N) \cdot \operatorname{diam}(Q)^{n-1}}{4A^n} \\
\leq\ & \frac{1}{2}F(N) \cdot \operatorname{diam}(Q)^{n-1} + \frac{1}{4}F(N) \cdot \operatorname{diam}(Q)^{n-1} \\
\leq\ & \frac{3}{4}F(N) \cdot \operatorname{diam}(Q)^{n-1},
\end{aligned}$$

which contradicts (4.1).

Now we know that if $N \geq N_0(n)$, then

$$F(N) \leq 4A \cdot F\left(\frac{N}{1+c}\right).$$

- If $1 \leq N \leq e$, then the Hardt-Simon bound
$$C_1 e^N \leq C_1 e^e.$$

- If $e < N \le N_0$, then the Hardt-Simon bound
$$C_1 e^N \le N^\alpha,$$
by choosing $\alpha = (N_0 + \log C_1)/\log N_0 \ge (N + \log C_1)/\log N$ since $t/\log t$ is increasing when $t > e$.
- If $N \ge N_0$, then let
$$k = \left\lfloor \log_{(1+c)} \frac{N}{N_0} \right\rfloor + 1 \ge \log_{(1+c)} \frac{N}{N_0}.$$
So $(1+c)^k \ge N/N_0$, i.e.,
$$\frac{N}{(1+c)^k} \le N_0.$$
Therefore,
$$\begin{aligned} F(N) &\le 4A \cdot F\left(\frac{N}{1+c}\right) \\ &\le (4A)^2 \cdot F\left(\frac{N}{(1+c)^2}\right) \\ &\le \cdots \\ &\le (4A)^k \cdot F\left(\frac{N}{(1+c)^k}\right) \\ &\le (4A)^k \cdot \exp\left(\frac{N}{(1+c)^k}\right) \\ &\le (4A)^{\log_{(1+c)} \frac{N}{N_0} + 1} \cdot e^{N_0} \\ &\le 4Ae^{N_0} N_0^{-\log_{(1+c)}(4A)} \cdot N^{\log_{(1+c)}(4A)}. \end{aligned}$$

Choose
$$C = \max\left\{C_1 e^e, 4Ae^{N_0} N_0^{-\log_{(1+c)}(4A)}\right\} \quad \text{and} \quad \alpha = \max\left\{\frac{N_0 + \log C_1}{\log N_0}, \log_{(1+c)}(4A)\right\}.$$
We then have that
$$F(N) \le CN^\alpha,$$
and the theorem is complete. ☐

Remark. Seeing the proofs of Theorems 3.21 and 4.1, it is hopeless to derive a useful estimate of the exponent α, even for harmonic functions in $\mathbb{R}^n$.

As an immediate concequence,

THEOREM 4.2 (Upper bound of the nodal size of harmonic functions). *There exist positive constants $\alpha = \alpha(n)$, $C = C(n)$, and $K = K(n)$ such that*
$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x,r)) \le C \cdot r^{n-1} N_u(x, Kr)^\alpha$$
for all harmonic functions u in $\mathbb{B}(x, 2Kr)$.

PROOF. Let Q be the cube with center x and side-length $2r$. Then $B(x,r) \subset Q$ and
$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x,r)) \le C_1 \cdot \operatorname{diam}(Q)^{n-1} N_u(Q)^\alpha = C_1(2\sqrt{n})^{n-1} \cdot r^{n-1} N_u(Q)^\alpha,$$

in which $C_1 = C_1(n)$ and $\alpha = \alpha(n)$ are from Theorem 4.1. Recall that

$$N_u(Q) = \sup_{y\in Q, s\in(0,\mathrm{diam}(Q))} N_u(y,s).$$

By Theorem 3.15, for $y \in Q \subset \mathbb{B}(x, \sqrt{n}r)$ and $0 < s < \mathrm{diam}(Q) = 2\sqrt{n}r$,

$$N_u(y,s) \leq CN_u(x, Kr),$$

for some positive constants C and K. The theorem is complete. □

4.2. Nodal size of Laplacian eigenfunctions

In this section, we prove a simplified version of the nodal size estimates of Laplacian eigenfunction in Logunov [**Lo1**].

THEOREM 4.3 (Upper bound of the nodal size of Laplacian eigenfunctions). *Let $R > 0$. There exist positive constants $\alpha = \alpha(n)$, $C = C(R,n)$, and $K = K(n) \geq 1$ such that if $0 < r \leq R/K$ and $\max_{\mathbb{B}(x_0,R)} |\phi| = |\phi(x_0)|$, then*

$$\mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_0, r)) \leq C \cdot r^{n-1}\lambda^\alpha$$

for all Laplacian eigenfunctions ϕ, $-\Delta\phi = \lambda\phi$, in $\mathbb{B}(x_0,R)$.

Remark. Comparing with the nodal size estimate of harmonic functions in Theorem 4.2, we have additional conditions:

- We introduce $R > 0$ and require $r \leq R/K$.
- We require that $|\phi|$ achieves its maximum in $B(x_0,R)$ at x_0.

We therefore conclude that the upper bound $C \cdot r^{n-1}\lambda^\alpha$ contains the exponent $\alpha = \alpha(n)$ as a dimensional constant and the coefficient $C = C(R,n)$ that also depends on the size R of the domain.

Remark (Theorem 4.3 on manifolds). The Laplacian on a Riemannian manifold is a second order elliptic partial differential operator. Logunov [**Lo1**] asserts that the same upper bound in Theorem 4.3 for Laplacian eigenfunctions on manifolds. To achieve this full version of the theorem, one would have to establish the nodal size estimates of solutions to general elliptic PDEs (rather than just for harmonic functions in the previous section). In terms of the two additional conditions:

- The condition that $R > 0$. This condition is harmless when applying to Laplacian eigenfunctions on compact manifolds $\mathbb{M}$. Indeed, one can simply choose $R = \mathrm{diam}(\mathbb{M}) < \infty$ since $\mathbb{M}$ is compact.
- The condition that $|\phi|$ achieves maximum in $B(x_0,R)$ at x_0. This condition is also harmless when applying to Laplacian eigenfunctions on compact manifolds $\mathbb{M}$. Indeed, one can simply choose $x_0 \in \mathbb{M}$ such that $|\phi(x_0)| = \max_{\mathbb{M}} |\phi|$ since $\mathbb{M}$ is compact[i].

As mentioned in the Preface, the framework of [**Lo1**] is same as the one presented in this note, and we refer to his original paper for details.

[i]The rigorous proof requires a chain of balls to connect any point on $\mathbb{M}$ with x_0 that is similar to the one used to prove the Harnack's inequality in Theorem 2.10. See Logunov-Malinnikova [**LM2**, Proposition 2.4.2] for details.

The idea to prove the nodal size estimates of eigenfunctions $\phi(x)$ has been mentioned in the beginning of Chapter 2: Let $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$. Then u is harmonic in $B(x_0, Kr) \times \mathbb{R}$ and their nodal size are related in a simple fashion. In particular,

$$\mathcal{H}^{n-1}(\mathcal{N}(\phi) \cap B(x_0,r)) = \frac{1}{2r} \cdot \mathcal{H}^n\left(\mathcal{N}(u) \cap B(x_0,r) \times [-r,r]\right).$$

We then need to estimate the doubling index of u in terms of λ.

Lemma 4.4. *Let $R > 0$ and $-\Delta\phi = \lambda\phi$ in $\mathbb{B}(x_0,R) \subset \mathbb{R}^n$. Suppose that $\max_{\mathbb{B}(x_0,R)} |\phi| = |\phi(x_0)|$. Write $u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$ and $p_0 = (x_0, 0) \in \mathbb{R}^{n+1}$. Then there exist positive constant $C = C(R)$ such that*

$$N_u(p_0, r) \le C\sqrt{\lambda} \quad \text{for all } 0 < r \le \frac{R}{2}.$$

Proof. Since $\max_{\mathbb{B}(x_0,R)} |\phi| = |\phi(x_0)|$,

$$\max_{\mathbb{B}(p_0,r)} |u(x,t)| = e^{r\sqrt{\lambda}}|\phi(x_0)|$$

for all $0 < r \le R$. Hence,

$$N_u(p_0,r) = \log_2\left(\frac{\max_{\mathbb{B}(p_0,2r)} |u|}{\max_{\mathbb{B}(p_0,2r)} |u|}\right) \le \log_2\left(\frac{e^{2r\sqrt{\lambda}}|\phi(x_0)|}{e^{r\sqrt{\lambda}}|\phi(x_0)|}\right) = \log_2 e \cdot r\sqrt{\lambda} \le C\sqrt{\lambda},$$

by choosing $C = C(R) = (\log_2 e) \cdot R$. □

Now we prove Theorem 4.3.

Proof of Theorem 4.3. Continue with the same notations as above. By Theorem 4.2, there are constants $\alpha_1 = \alpha_1(n)$, $C_1 = C_1(n)$, and $K_1 = K_1(n)$ such that

$$\mathcal{H}^n\left(\mathcal{N}(u) \cap B^{n+1}\left(p_0, \sqrt{n}r\right)\right) \le C_1\left(\sqrt{n}r\right)^n N_u\left(p_0, \sqrt{n}K_1 r\right)^{\alpha_1}.$$

Choose $K = K(n) = 2\sqrt{n}K_1$. Then $\sqrt{n}K_1 r = Kr/2 \le R/2$ if $r \le R/K$. By the above lemma,

$$N_u(p_0,r) \le C_2\sqrt{\lambda}$$

for some $C_2 = C_2(R)$. Therefore, by the subadditivity of the Hausdorff measure,

$$\begin{aligned}
\mathcal{H}^{n-1}\left(\mathcal{N}(\phi) \cap B^n(x_0,r)\right) &= \frac{1}{2r} \cdot \mathcal{H}^n\left(\mathcal{N}(u) \cap B^n(x_0,r) \times [-r,r]\right) \\
&\le \frac{1}{2r} \cdot \mathcal{H}^n\left(\mathcal{N}(u) \cap B^{n+1}\left(p_0, \sqrt{n}r\right)\right) \\
&\le \frac{1}{2r} C_1(\sqrt{n}r)^n N_u\left(p_0, \sqrt{n}K_1 r\right)^{\alpha_1} \\
&\le \frac{n^{\frac{n}{2}} C_1 C_2^{\alpha_1}}{2} \cdot r^{n-1}\lambda^{\frac{\alpha_1}{2}} \\
&= C \cdot r^{n-1}\lambda^{\alpha},
\end{aligned}$$

by choosing $\alpha = \alpha(n) = \alpha_1/2$ and $C = C(R,n) = n^{\frac{n}{2}} C_1 C_2^{\alpha_1}/2$. □

APPENDIX A

Nodal sets of harmonic polynomials

In Sections 3.1 and 3.3, we use polynomials as the simplest examples to study the nodal sets. In particular, with the help of fundamental theorem of algebra, the upper bound of the nodal size of polynomials is immediate: For any polynomial P in $\mathbb{R}^n$,

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B(x,r)) \leq C \cdot r^{n-1} \deg P,$$

in which $\deg P$ is the degree of P and one can take the constant $C = n\alpha_{n-1}$ (which is possibly not sharp). In this chapter, we continue to use polynomials to study the nodal sets, with the focus now on the lower bound.

A.1. Lower bound of the nodal size: Take II

Recall the lower bound of the nodal size of harmonic functions in Conjecture 3.4: There is a positive constant $c = c(n)$ such that

$$\mathcal{H}^{n-1}(\mathcal{N}(u) \cap B(x_0,r)) \geq c \cdot r^{n-1}$$

for all harmonic functions u in $B(x_0, r)$ with $u(x_0) = 0$. Other than Logunov's proof for all harmonic functions in [**Lo2**], it is not known if there is an alternative and easier approach for harmonic polynomials. Therefore, it is interesting to ask

Question (Lower bound of the nodal size of harmonic polynomials). Prove that

$$\mathcal{H}^{n-1}(\mathcal{N}(P) \cap B(x_0,r)) \geq c \cdot r^{n-1}$$

for all harmonic polynomials P such that $P(x_0) = 0$.

Without the harmonic condition, there can be no non-trivial lower bound. For example, let $u(x_1, ..., x_n) = x_1^2 + \cdots + x_n^2$. Then u only vanishes at the origin and its nodal size in the $(n-1)$-dim measure is zero.

APPENDIX B

Suggested problems for presentation

Students are encouraged to give short presentations in the course. Here is a list of suggested problems for presentation.

B.1. Homework

You can present one of the following problems from the homework assignments.

- Problem 2-1.
- Problem 2-2.
- Problem 2-3.
- Problem 3-4.
- Problem 3-5.

B.2. Yau's problem section

Conducting literature research is one of the basic academic skills. The following problems are selected from Yau's influential problem set [**Y**], most of which are still open today. You are encouraged to do literature research about one of the problems, with the focus on the background, history, and current state of art. In particular,

(1). Read the original text in Yau [**Y**]. If there are terminology or notations with which you are not familiar[i], then find resources for them. Wikipedia is always the most direct resource, which also contains reference of books and articles for additional materials.

(2). Some problems are so famous that there are Wikipedia entries for them, e.g., the question of "Can one hear the shape of a drum" and related Problems 67, 68 and 69 in Yau [**Y**]. From Wikipedia and the references, you can find most of the information about the background, history, and current state of art.

(3). The most useful mathematics literature database is MathSciNet maintained the American Mathematical Society. You should be able to access it on campus and through university portal.

Here are some information about the problems for you to start the literature research.

- Problems 67, 68 and 69. This group of problems are all related to the old question of "Can one hear the shape of a drum" popularized by Kac[ii], i.e., whether the geometry of a manifold ("shape") can be determined by the Laplacian eigenvalues ("pitches").
- Problems 70. This problem was conjectured by Pólya[iii]: Let Ω be a bounded domain in $\mathbb{R}^2$ and λ_j be the j-th Dirichlet Laplacian eigenvalue (counting with multiplicity). Then
$$\lambda_j \geq \frac{4\pi j}{\text{Area}(\Omega)}.$$

[i] For example, "isometric" in Problems 67 and 69, "tile" in Problem 70, "genus" in Problem 75, "critical points" in Problem 76.

[ii] M. Kac, *Can one hear the shape of a drum*? Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23.

[iii] G. Pólya, *On the eigenvalues of vibrating membranes.* Proc. London Math. Soc. (3) 11 1961 419–433.

In the same paper, Pólya proved the conjecture when the domain Ω tiles $\mathbb{R}^2$ (i.e., the translations of Ω fill $\mathbb{R}^2$, examples of such domains include the squares.) The conjecture remains open for general domains. The best known result is due to Li-Yau[i]:

$$\lambda_j \geq \frac{2\pi j}{\mathrm{Area}(\Omega)}.$$

- Problem 74. This problem is the nodal size conjecture 1.1 discussed in this note!
- Problem 75. Cheng[ii] proved that the multiplicity of j-th eigenvalue on a compact surface with genus g (which is a topological quantity related to the number of holes of the surface) is bounded by

$$\frac{(2g+j+1)(2g+j+2)}{2}.$$

This problem asks whether similar result holds in higher-dimensions, i.e. whether the multiplicity of eigenvalues can be controlled by certain topological information of the manifold of dimension greater than two.
- Problem 76. Let ϕ_j be the j-th eigenfunction in a domain $\Omega \subset \mathbb{R}^2$. A point $x \in \Omega$ is called a critical point of ϕ_j if $\phi_j(x) = 0$ and $\nabla\phi_j(x) = 0$. One can show that the set of critical points is discrete. This problem asks whether the number of critical points grows as $j \to \infty$. The answer to this problem is in general negative, see Jakobson-Nadirashvili[iii] and a recent result by Buhovski-Logunov-Sodin[iv].
- Problem 77. This problem was conjectured by Payne-Pólya-Weinberger[v] and is usually referred as the eigenvalue ratio conjecture: Let Ω be a bounded domain in $\mathbb{R}^2$ and $\lambda_j(\Omega)$ be the j-th Dirichlet Laplacian eigenvalue on Ω. Then

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B(0,1))}{\lambda_1(B(0,1))}.$$

Furthermore, if the equality holds, then Ω is a disk. This conjecture has been solved by Ashbaugh-Benguria[vi].
- Problem 78. This problem was conjectured by Payne[vii]: Let Ω be a bounded and convex domain in $\mathbb{R}^2$. We know that the first Dirichlet eigenfunction does not change sign in Ω, and the second eigenfunction ϕ_2 has non-empty nodal set in Ω. This problem then asks whether the nodal curve of ϕ_2 can form a closed loop in Ω. This conjecture has been proved by Melas[viii], which also contains several earlier results. Without the

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convexity condition, the conjecture may fail according to the example constructed by Hoffmann-Ostenhof, Hoffmann-Ostenhof, and Nadirashvili[i].

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