Dimensions of Projections & level sets

## 1. Fourier techniques:

Recall, for $A \subseteq \mathbb{R}^n$ cpt, $\dim_H(A) = \sup\{s : \exists \mu \in \mathcal{M}(A) \text{ s.t. } I_s(\mu) < \infty\}$

Thm: For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $0 < s < n$, $\quad I_s(\mu) = \gamma(n,s) \int |\hat{\mu}(x)|^2 |x|^{s-n} \, dx$.

For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $\hat{\mu}(\xi) := \int e^{-2\pi i x \cdot \xi} \, d\mu(x)$. Let $k_s(x) = |x|^{-s}$.

Idea of pf. of thm.: $I_s(\mu) = \iint |x-y|^{-s} \, d\mu(x) \, d\mu(y) = \int (k_s * \mu)(y) \, d\mu(y) = \int \widehat{(k_s * \mu)}(\xi) \, \bar{\hat{\mu}}(\xi) \, d\xi$

$= \int \widehat{k_s}(\xi) \hat{\mu}(\xi) \bar{\hat{\mu}}(\xi) \, d\xi = \int \widehat{k_s}(\xi) |\hat{\mu}(\xi)|^2 \, d\xi$.

$k_s(x) \notin L^1 + L^2$ but is a tempered distribution.

Lemma: $\widehat{k_s}(\xi) = \gamma(n,s) |\xi|^{s-n}$ as tempered distributions.

$\Rightarrow \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \int k_s \hat{\varphi} = \int \gamma(n,s) k_{n-s} \varphi$.

Pf: For $n/2 < s < n$, $k_s \in L^1 + L^2 \Rightarrow \widehat{k_s} \in L^\infty + L^2$, $\quad k_s(rx) = r^{-s} k_s(x) \Rightarrow \widehat{k_s}(r\xi) = r^{s-n} \widehat{k_s}(x)$

As the Fourier transform of a radial fn. is radial, we must have $\widehat{k_s}(\xi) = \gamma(n,s) k_{n-s}(\xi)$.

For $0 < s < n/2$, do it by duality. $\int \widehat{k_s} \varphi = \int k_s \hat{\varphi} = \int \gamma(n, n-s)^{-1} \widehat{k_{n-s}} \hat{\varphi} = C \int k_{n-s} \hat{\hat{\varphi}} = C \int k_{n-s}(x) \varphi(-x)$.

For $s = n/2$, take limits. ($\triangle$ Need to show $\forall n$ $\lim_{s \to n/2} \gamma(n,s) = 1$)

Pf of thm: Step 1: $\quad d\mu = \varphi \, dx \quad$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$I_s(\mu) = \iint \underbrace{|x-y|^{-s} \varphi(x)}_{\cancel{\in \mathcal{S}(\mathbb{R}^n)}} \underbrace{\varphi(y)}_{\in \mathcal{S}(\mathbb{R}^n)} \, dx \, dy = \cancel{\int \widehat{k_s * \varphi}(\xi) \bar{\hat{\varphi}}(\xi) \, d\xi} = \cancel{\int \widehat{k_s}(\xi) |\hat{\varphi}(\xi)|^2 \, d\xi}$.

Let $z = y - x$

$= \iint |z|^{-s} \varphi(y-z) \varphi(y) \, dz \, dy = \iint |z|^{-s} \tilde{\varphi}(z-y) \varphi(y) \, dy \, dz = \int |z|^{-s} \underbrace{(\tilde{\varphi} * \varphi)}_{\in \mathcal{S}(\mathbb{R}^n)}(z) \, dz$

Note $\tilde{\varphi} * \varphi = \widehat{|\hat{\varphi}|^2} \Rightarrow \quad = \int \gamma(n,s) |z|^{n-s} |\hat{\varphi}|^2(z) \, dz$.

Step 2: Let $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi_\varepsilon(x) = \varepsilon^{-n} \psi(\frac{x}{\varepsilon})$, $\quad \mu_\varepsilon = \psi_\varepsilon * \mu = f_\varepsilon \, dx$, $f_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$

Dimension of Level Sets : Upper bounds

1. Hölder fns.: A co-area formula

Thm : Let $f \in C^\alpha(\mathbb{R}^n)$, $A \subseteq \mathbb{R}^n$ Borel. For every $\beta \geq 0$ $\exists C(\alpha,\beta)$ st.

$$\int \mathcal{H}^\beta(f^{-1}(\{t\}) \cap A)\,dt \leq C\,[f]_\alpha\, \mathcal{H}^{\alpha+\beta}(A).$$

Pf (due to J. Hirsch): Cover $A \subseteq \bigcup_j B^i_j$ with balls st. $\mathrm{diam}(B^i_j) < \frac{1}{i}$

$$\sum_j \mathrm{diam}(B^i_j)^{\alpha+\beta} \leq c\,\mathcal{H}^{\alpha+\beta}_{i^{-1}}(A) + \frac{1}{i}.$$

Let $g^i_j(y) := \mathrm{diam}(B^i_j)^\beta\, \mathbb{1}_{f(B^i_j)}(y) \geq 0$. So we can define $g^i(y) = \sum_j g^i_j(y)$

Now $\int g^i(y)\,dy = \sum_j (\mathrm{diam}(B^i_j))^\beta \mathcal{L}^1(\cancel{B}f(B^i_j)) \leq \sum (\mathrm{diam}(B^i_j))^{\beta+\alpha} \leq C\,\mathcal{H}^{\alpha+\beta}_{i^{-1}}(A) + \frac{1}{i}$

But note $\mathcal{H}^\beta_{i^{-\alpha}}(A \cap f^{-1}(\{y\})) \leq g^i(y)$. Taking $i \nearrow \infty$ gives the result. $\square$

# COMPUTING THE DIMENSION OF MEASURES FROM THEIR PROJECTION

0.0.1. *Question.* Consider $\mathbb{R}^d$ and the projection map $\pi : \mathbb{R}^d \to \mathbb{R}^m$ that projects to the first $m$ coordinates. Consider also a non-negative, Borel measure $\mu$ on $\mathbb{R}^d$ with $\operatorname{supp}\mu$ compactly-supported. For a positive real number $s$, the $s$-capacity of $\mu$ is defined as

$$I_s(\mu) := \iint \frac{1}{|x-y|^s}\, d\mu(y) d\mu(x)\,.$$

The dimension of $\mu$ is defined as

$$\dim \mu := \sup_s \{s \in \mathbb{R} : I_s(\mu) < \infty\}\,.$$

We wish to compute $\dim \mu$ from the "observable" projected measure $\pi_{\#}\mu$.

0.0.2. *Main theorem.* We work in the setting $m = d-1$ and $d-1 < \dim \mu < d$. Our main theorem is

**Theorem 0.1.** *Assuming $\mu$ is an ADR measure, we have that $\pi_{\#}\mu$ is absolutely continuous w.r.t. the Lebsgue measure $dx$ on $\mathbb{R}^{d-1}$ for a.e. projection $\pi$. Moreover, writing $\pi_{\#}\mu = f\,dx$, we have*

$$\text{Hölder-reg}(f) \leqslant \dim \mu - d + 1 \leqslant L^2\text{-reg}(f)$$

*Proof.* Using Plancherel and the convolution identity for Fourier transforms on the definition of the $s$-capacity, we have

$$\begin{aligned}
I_s(\mu) &:= \int \left(\widehat{\int |x-y|^{-s}\, d\mu(y)}\right) \widehat{d\mu(x)} \\
&= \int |k|^{s-d} |\widehat{\mu}(k)|^2\, dk
\end{aligned}$$

Now we use a polar decomposition of frequency space. Let $Gr(d, d-1)$ denote the Grassmannian of $(d-1)$-hyperplanes passing through the origin in $\mathbb{R}^d$. Let us denote such planes by $\theta$ and let $d\theta$ be the uniform measure on $Gr(d, d-1)$, i.e. the measure that is invariant under the action of the orthogonal group $O(d)$. Now

$$\begin{aligned}
\int |k|^{s-d} |\widehat{\mu}(k)|^2\, dk &= \int_{Gr(d,d-1)} \int_\theta |k|^{s-d} |\widehat{\mu}(k)|^2\, dk\, |k| d\theta \\
&= \int_{Gr(d,d-1)} \int_\theta |k|^{s-d+1} |\widehat{\mu}|_\theta(k)|^2\, dk\, d\theta
\end{aligned}$$

where the inner integral is over points $k \in \theta$ and the additional factor of $|k|$ comes from the change of variables. Thus if $I_s(\mu)$ is finite, we have that for almost every $\theta$

$$I_s(\mu, \theta) := \int_\theta |k|^{s-d+1} |\widehat{\mu}|_\theta(k)|^2\, dk < \infty\,.$$

Now as restriction in frequency space is the same as projection in real space, i.e.

$$\widehat{\mu}|_\theta = \widehat{\pi_{\#}\mu}\,,$$

where $\pi$ is the orthogonal projection onto the plane $\theta$, we have that for almost every projection $\pi$

$$I_s(\pi_{\#}\mu) = \int |k|^{s-d+1} |\widehat{\pi_{\#}\mu}(k)|^2 \, dk = \int_\theta |k|^{s-d+1} |\widehat{\mu}|_\theta(k)|^2 \, dk < \infty .$$

As we assumed $d-1 < \dim\mu$, we have that $I_s(\pi_{\#}\mu)$ is finite for some $s > d-1$ and so

$$\int |\widehat{\pi_{\#}\mu}(k)|^2 \, dk \leqslant \int |k|^{s-d+1} |\widehat{\pi_{\#}\mu}(k)|^2 \, dk < \infty .$$

So we have that $\widehat{\pi_{\#}\mu} \in L^2(\mathbb{R}^{d-1}, dk)$ and, so, by Parseval we have that $\pi_{\#}\mu \in L^2$. So, it is absolutely continuous w.r.t. $dx$ and we can write $\pi_{\#}\mu = f\,dx$.

Now we prove the upper bound. This follows simply from noting that $I_s(\pi_{\#}\mu)$ is simply the $(s-d+1)$-homogeneous Sobolev semi-norm of $f$:

$$I_s(\pi_{\#}\mu) = \int |k|^{s-d+1} |\widehat{\pi_{\#}\mu}(k)|^2 \, dk = \int |k|^{s-d+1} |\hat{f}(k)|^2 \, dk = \|f\|_{\mathring{H}^{s-d+1}} .$$

It remains to prove the lower bound. Here is where we use the assumption that $\mu$ is an ADR measure. Thus, there exists constants $c, C$ and $d-1 < \alpha < d$, such that for any $x \in \operatorname{supp}\mu, 0 < r < \operatorname{diam}(\operatorname{supp}\mu)$, we have

$$cr^\alpha \leqslant \mu(B_r(x)) \leqslant Cr^\alpha .$$

Now choose any point $p \in \partial(\operatorname{supp} f) \subset \mathbb{R}^{d-1}$. So, for any $r > 0$, there exist $q, q' \in B_r^{d-1}(p) \subset \mathbb{R}^{d-1}$ so that $f(q) = 0$ and $f(q') > 0$. Now we have

$$\begin{aligned} cr^\alpha \leqslant \mu(B_r(x)) \leqslant \mu(B_r^{d-1}(p) \times \mathbb{R}) = \pi_{\#}\mu(B_r^{d-1}(p)) &= \int_{B_r^{d-1}(p)} f \, dx \\ &\leqslant A \|f\|_{L^\infty(B_r^{d-1}(p))} r^{d-1} . \end{aligned}$$

So, there exist $q, q' \in \mathbb{R}^{d-1}$ satisfying $|q - q'| = r$ such that

$$|f(q) - f(q')| \geqslant \|f\|_{L^\infty(B_r^{d-1}(p))} \geqslant cA^{-1} r^{\alpha-d+1}$$

and so we see that $f$ is at most $(\alpha - d + 1)$-Hölder continuous. $\square$