DIMENSION OF LEVEL SETS OF SOBOLEV FUNCTIONS : UPPER BOUNDS

RECALL: If $f \in C^\alpha(\mathbb{R}^n)$, $A \subseteq \mathbb{R}^n$ Borel, $\forall \beta \geq 0$ $\exists C = C(\alpha, \beta)$ st

$$\int \mathcal{H}^\beta(f^{-1}(\{t\}) \cap A)\, dt \leq C[f]_\alpha \mathcal{H}^{\alpha+\beta}(A).$$

$\vdash$

Now $f \in W^{s,p}$, i.e.

$$[f]_{s,p}^p = \iint \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy < +\infty$$

Defn: $\Omega \subseteq \mathbb{R}^n$ be open, $f \in L^1(\Omega)$. Let $f_{x,r} := ⨍_{B_r(x)} f\, dy$ $(B_r(x) \subseteq \Omega)$. We define

$$L_t(f) := \{ x \in \Omega : \liminf_{r \to 0} f_{x,r} \leq t \leq \limsup_{r \to 0} f_{x,r} \}.$$

Thm: Let $\Omega \subseteq \mathbb{R}^n$ open, $0 < s < 1$, $1 \leq p < \infty$, $f \in W^{s,p}(\mathbb{R}^n)$

Then a.e. t, the Hausdorff dim of $L_t(f)$ is

- at most $n - \frac{sp}{p+1}$ when $s \leq \frac{n}{p}$
- at most $n - s + \frac{n}{p(p+1)}$ when $s > \frac{n}{p}$.

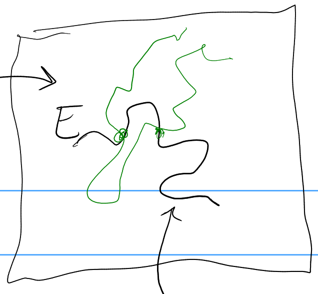
Rmk: We know (Sobolev embedding) $W^{s,p} \hookrightarrow C^\alpha$, $ns > p$, $\alpha < s - \frac{n}{p}$

$$\Rightarrow \dim_{\mathcal{H}}(L_t(u)) \leq n - \alpha = n - s + \frac{n}{p}$$

$$\rotatebox{90}{$\geq$}\ n - s + \frac{n}{p(p+1)} \quad <$$

Strategy of Pf.:

$f\in C^{s'}$



$f\in W^{s,p}$

$\forall\, 0\le s'\le s$

Have estimates on $\dim_{\mathcal{H}}(E)$

Prop: Let $\Omega\subseteq\mathbb{R}^n$ open, $f\in W^{s,p}(\Omega)$, $0<s<1$, $1\le p<\infty$. Fix $0\le s'\le s$, let

$$m_{s'}(x) := \sup_{0<r\le 1}\frac{1}{\omega_n r^{n+s'}}\int_{B_r(x)}|f(y)-f_{x,r}|\,dy.$$

Then we have:

1. $\forall\, 0<\lambda$, $E_\lambda := \Omega\cap\{m_{s'}\le\lambda\}$ is closed.
2. $\bar{f}(x) := \lim_{r\to 0} f_{x,r}$ exists for every $x\in E_\lambda$

   $f_\lambda := \bar{f}|_{E_\lambda}\in C^{s'}$, $[f_\lambda]_{s'}\le C(s,n)\lambda$
3. If $s'=s$, then $|\Omega\cap\{m_s>\lambda\}|\lesssim_\Omega \lambda^{-p}[f]^p_{s,p}$

   In particular, $\{m_s=\infty\}$ has zero Lebesgue measure.
4. For $s'<s$,

$$\mathcal{H}^{n-(s-s')p}(\Omega\cap\{m_{s'}=\infty\})=0.$$

$\vdash$

Pf of thm given prop:

$$\dim_{\mathcal{H}}(L_t(f)) = \max\{\dim_{\mathcal{H}}(L_t(f)\cap\{m_{s'}<\infty\}),\ \dim_{\mathcal{H}}(L_t(f)\cap\{m_{s'}=\infty\})\}$$

$$\le \max\{n-s',\ n-(s-s')p\}$$

On optimizing s' $\Rightarrow$ $\dim_{\mathcal{H}}(L_t(f))\le n-\frac{sp}{p+1}$

For the second part, $s > \frac{n}{p}$, $f \in W^{s,p} \cap C^\alpha$, $\alpha = s - \frac{n}{p}$

$$\dim_H(L_t(f)) \leq \max\{n - s', \dim_H(L_t(f) \cap \underbrace{\{m_{s'} = \infty\}}_{=A})$$

For a.e. t, $\dim(L_t(f) \cap A) \leq \beta$

$$= n - (s-s')p - \left(s - \frac{n}{p}\right)$$

$\alpha + \beta = \dim_H A$
$= n - (s-s')p$
$\Rightarrow$
$\beta = n - (s-s')p$
$- s + \frac{n}{p}$

Again optimizing, we get the desired result.

$\square$

To prove the Prop, we'll need:

1. Vitali Covering Lemma: If $A \subseteq \mathbb{R}^n$ some set
$A \subseteq \bigcup_{i \in I} B_i$, $B_i \subseteq \mathbb{R}^n$ a ball of radius $r_i \leq 1$

Then $\exists I' \subseteq I$ st. $B_i \cap B_j = \phi$ for $i, j \in I'$

and Let $5B_i$ denote the ball with same center, radius $5r_i$

then $A \subseteq \bigcup_{i \in I'} 5B_i$. (see Wikipedia)

2. Basic Morrey-Campanato estimates: Let $f: \Omega \to \mathbb{R}$
$f \in L^1(\Omega)$: $\mathcal{S} := \sup_{x,\, 0 < r \leq 1} r^{-n-s} \int_{B_r(x)} |f - f_{x,r}|\, dy < \infty$

a $\Rightarrow$ For $x, y \in \Omega$, $|f_{x,r} - f_{y,r}| \leq C\mathcal{S}|x-y|^s$
$r \sim |x-y|$

b $\Rightarrow$ If $x \in \Omega$, $0 < \frac{r_1}{2} \leq r_2 \leq r_1 \leq 1$, then
$$|f_{x,r_1} - f_{x,r_2}| \leq C\mathcal{S} r_1^s.$$

To show these

$$r^{-s}|f_{x,r} - f_{y,r}| = r^{-s}\left| ⨍_{B_r(x)} f-c - ⨍_{B_r(y)} f-c \right|$$

$C \sim ⨍_{B_{3r}(z)} f$. Complete the pf.! $\square$

$\vdash$

Pf. of Prop: Note $x \mapsto r^{-n-s'} \int_{B_r(x)} |f(y) - f_{x,r}|\, dy$ is continuous $\Rightarrow m_{s'}$ is lower semi-cts (as it is a sup of cts fns.) $\Rightarrow \{m_{s'} \le \lambda\}$ is closed, proving 1.

To prove 2. $\overset{(b)}{\Rightarrow}$ • $f_{x,2^{-n}}$ is Cauchy and

$$f_{x,2^{-n}} \to \bar{f}(x).$$

$$\bullet \quad \lim_{r\to 0} f_{x,r} = \bar{f}(x)$$

$$\bullet \quad |f_{x,r} - \bar{f}(x)| \le CSr^{s'}$$

Combining with (a), we get $\bar{f} \in C^{s'}$ + the semi-norm estimate. Proving 2.

To prove 3, note

$$(*) \quad m_{s'}(x) = \sup_{0<r\le 1} \frac{1}{r^{s'}} ⨍_{B_r(x)} |f - f_{x,r}|\, dy$$

$$\le \sup_{0<r\le 1} \frac{1}{r^{s'}} \left[⨍_{B_r(x)} |f - f_{x,r}|^p\, dy \right]^{1/p}$$

Used Jensen, $⨍|f| \le \left(⨍|f|^p\right)^{1/p}$

Let $m_{s',p}(x) := \sup_{0<r\le 1} \frac{1}{r^{s'p}} ⨍_{B_r(x)} |f - f_{x,r}|^p$

$(*)$ says $m_{s',1} \leq m_{s',p}^{1/p}$.

Now, $m_{s',p} = \sup \frac{1}{r^{s'p}} \fint_{B_r(x)} \left| f(y) - \fint_{B_r(x)} f\,dz \right|^p dy$

$$= \sup \frac{1}{r^{s'p}} \fint_{B_r(x)} \left| \fint_{B_r(x)} f(y) - f(z)\, dz \right|^p dy$$

$(**)$
$$\leq \sup \frac{1}{r^{s'p}} \fint_{B_r(x)} \fint_{B_r(x)} |f(y)-f(z)|^p\, dz\, dy$$

$$= \sup \frac{1}{\omega_n^2 \cancel{r^2} r^{2n+s'p}} \int_{B_r(x)\times B_r(x)} |f(y)-f(z)|^p\, dz\, dy$$

$$\leq \sup \frac{1}{\omega_n^2 r^{n+(s'-s)p}} \int_{B_r(x)\times B_r(x)} \frac{|f(y)-f(z)|^p}{|y-z|^{n+sp}}\, dy\, dz$$

$(y,z) \in B_r(x)\times B_r(x)$, $|y-z| < r$

Let us define,

$$M_{s',p}(x) := \sup_{0<r\leq 1} \frac{1}{\omega_n^2 r^{n-(s-s')p}} \int_{B_r(x)\times B_r(x)} \frac{|f(y)-f(z)|^p}{|y-z|^{n+sp}}\, dy\, dz$$

$(*)$ + $(**)$ says $m_{s'} \leq (M_{s,p})^{1/p}$

So, we have, $\{m_{s'} > \lambda\} \subseteq \underbrace{\{M_{s,p} > \lambda^p\}}_{G_\lambda}$

$x \in G_\lambda$, pick $r(x)$ s.t

$$\underbrace{C \int_{B_{r(x)}(x)\times B_{r(x)}(x)} \frac{|f(y)-f(z)|^p}{|y-z|^{n+sp}}\, dy\, dz}_{} \geq \lambda^p r(x)^{n-(s-s')p}$$

$$\geq \lambda^p r(x)^n$$

$G_\lambda \subseteq \bigcup_{x \in G_\lambda} B_{r(x)}(x)$, By Vitali, $\exists\, G_\lambda' \subseteq G_\lambda$

$$B_{r(x)}(x) \cap B_{r(y)}(y) = \phi \quad \forall x \neq y \in G_\lambda'$$

and $\quad G_\lambda \subseteq \bigcup_{x \in G_\lambda'} B_{5r(x)}(x)$

So,

$$|G_\lambda| \leq \sum_{x \in G_\lambda'} \omega_n (5r(x))^n \leq \lambda^{-p} 5^n C \sum_{x \in G_\lambda'} \int_{B_{r(x)}(x) \times B_{r(x)}(x)} (\cdots)$$

$$\leq \lambda^{-p} 5^n C \int_{\Omega \times \Omega} \frac{|f(y) - f(z)|^p}{|y - z|^{n+sp}} \, dy\, dz < +\infty$$

$$\lesssim \lambda^{-p} [f]_{s,p}^p$$

Proving 3.

To prove 4. $\quad G_\lambda \subseteq \bigcup_{x \in G_\lambda'} B_{5r(x)}(x)$

$$\sum_{x \in G_\lambda'} (\operatorname{diam}(B_{r(x)}(x)))^{n - p(s - s')} \leq C \lambda^{-p} [f]_{s,p}^p$$

Note $\quad \{m_{s'} = \infty\} \subseteq \{M_{s,p} > \lambda^p\} = G_\lambda$

$$\mathcal{H}_\infty^{n-(s-s')p}(\{m_{s'} = \infty\}) \leq C \lambda^{-p} [f]_{s,p}^p$$

Letting $\lambda \nearrow \infty$, $\mathcal{H}_\infty^{n-(s-s')p}(\{m_{s'} = \infty\}) = 0$

$\mathcal{H}_\infty^\alpha(A) = 0 \Rightarrow \mathcal{H}^\alpha(A) = 0$ proving 4. $\square$