DIMENSION OF LEVEL SETS FOR RANDOM WAVELET SERIES

Consider $\mathbb{R}^n$ and let $\{\psi^i_{k,l}\}$ be Schwartz functions (normalized in L^∞) that form a wavelet basis for $L^2(\mathbb{R}^n)$. A function $f = \sum_{k=0}^\infty c^i_{k,l}\psi^i_{k,l}$ is a random wavelet series if the $c^i_{k,l}$ are a sequence of (real-valued) random variables. Our function f will possess some regularity given by a parameter α and we will denote by $\tilde{c}^i_{k,l} = 2^{k\alpha}c^i_{k,l}$. A random wavelet series is

- called a self-similar random process if the $C_k = \{\tilde{c}^i_{k,l}\}_{i,l}$ are identically distributed in $k \in \mathbb{N}$.
- said to have independent increments if $\{\tilde{c}^i_{k,l}\}_{i,k,l\in I}$ are independent whenever the corresponding $\{\psi^i_{k,l}\}_{i,k,l\in I}$ have pairwise disjoint supports.
- said to have stationary increments if all the $\tilde{c}^i_{k,l}$ share the same law.

1. Assumptions

Note that $\hat{F}(0) = 1$ and $\|\hat{F}\|_\infty = 1$. Let's assume the following:

(1) The random wavelet series we consider have independent stationary increments and we denote the common law shared by the $\tilde{c}^i_{k,l}$ by F.
(2) Fourier inversion holds on F; in particular, $F, \hat{F} \in L^1 \cap C^0$.
(3) $\hat{F} \geqslant 0$ and $\hat{F}$ has "good" behaviour near the origin.
(4) There exists a $c > 0$ so that for any $k \in \mathbb{N}, x \in \mathbb{R}^n$ ther exists i, l such that $|\psi^i_{k,l}(x)| \geqslant c$.

2. Preliminaries

Note first that $\psi^i_{k,l}(x) = \psi^i(2^kx+l)$. Since the wavelet expansion of f is absolutely convergent a.e., for $t, s \in \mathbb{R}^n$, $f(t) - f(s) = \sum c^i_{k,l}(\psi^i_{k,l}(t) - \psi^i_{k,l}(s))$ and so by independence

$$\mathbb{E}(e^{i(\xi_1 f(t) - \xi_2 f(s))}) = \prod \mathbb{E}(e^{ic^i_{k,l}(\xi_1\psi^i_{k,l}(t) - \xi_2\psi^i_{k,l}(s))}) =: \prod \phi^i_{k,l}(t, s, \xi_1, \xi_2)\,.$$

By scaling,

$$\begin{aligned}
\phi^i_{k,l}(t, s, \xi_1, \xi_2) &= \mathbb{E}(e^{ic^i_{k,l}(\xi_1\psi^i_{k,l}(t) - \xi_2\psi^i_{k,l}(s))}) = \mathbb{E}(e^{i\tilde{c}^i_{k,l}(2^{-k\alpha}\xi_1\psi^i(2^kt+l) - 2^{-k\alpha}\xi_2\psi^i(2^ks+l))}) \\
&= \int_{-\infty}^{\infty} e^{ix(2^{-k\alpha}\xi_1\psi^i(2^kt+l) - 2^{-k\alpha}\xi_2\psi^i(2^ks+l))}F(x)\,dx \\
&= \hat{F}(2^{-k\alpha}(\xi_1\psi^i(2^kt+l) - \xi_2\psi^i(2^ks+l)))
\end{aligned}$$

3. A bound

We note that

$$|\mathbb{E}(e^{i(\xi_1 f(t) - \xi_2 f(s))})| \leqslant \left| \prod_{2^{-k}\sim|t-s|} \phi^i_{k,l}(t, s, \xi_1, \xi_2) \right|\,.$$

In this case,

$$\begin{aligned}
\phi^i_{k,l}(t, s, \xi_1, \xi_2) &= \mathbb{E}(e^{ic^i_{k,l}(\xi_1\psi^i_{k,l}(t) - \xi_2\psi^i_{k,l}(s))}) = \mathbb{E}(e^{i\tilde{c}^i_{k,l}2^{-k\alpha}\xi_1\psi^i(2^kt+l)}) \\
&= \int_{-\infty}^{\infty} e^{ix2^{-k\alpha}\xi_1\psi^i(2^kt+l)}F(x)\,dx \\
&= \hat{F}(2^{-k\alpha}\xi_1\psi^i(2^kt+l))
\end{aligned}$$

So we have

$$|\mathbb{E}(e^{i(\xi_1 f(t)-\xi_2 f(s))})| \leqslant \left| \prod_{2^{-k}<|t-s|\leqslant 2^{-k-1}} \hat{F}(2^{-k\alpha}\xi_1\psi^i(2^k t+l))\hat{F}(2^{-k\alpha}\xi_2\psi^i(2^k s+l')) \right|$$

Thus

$$\int |\mathbb{E}(e^{i(\xi_1 f(t)-\xi_2 f(s))})|\, d\xi_1 d\xi_2 \leqslant C(t)C(s)2^{2k\alpha} \lesssim |t-s|^{-2\alpha}$$

where $C(t) = \int |\hat{F}(\xi_1\psi^i(2^k t+l))|\, d\xi_1 \sim \frac{\|\hat{F}\|_{L^1}}{|\psi^i(2^k t+l)|} < c^{-1}\|\hat{F}\|_{L^1}$.

4. Kahane's argument

Let $\delta_1 : \mathbb{R}^n \to [0,1]$ be a smooth compactly-supported function with $\delta_1(0) = 1$. Let $\delta_\varepsilon(x) = \varepsilon^{-d}\delta_1(x/\varepsilon)$. Write this using the Fourier transform as

$$\delta_\varepsilon(x) = \int \gamma(\varepsilon\xi)e^{i\xi x}\, d\xi\,.$$

Now consider

$$\delta_\varepsilon(f(x)) = \int \gamma(\varepsilon\xi)e^{i\xi f(x)}\, d\xi$$

and the measure $\mu_\varepsilon := \delta_\varepsilon(f(x))\,\mathcal{L}^n$. The capacity of this measure wrt. some kernel κ is given by

$$I(\mu_\varepsilon,\kappa) = \int \gamma(\varepsilon\xi_1)\bar{\gamma}(\varepsilon\xi_2)e^{i(\xi_1 f(t)-\xi_2 f(s))}\kappa(t-s)\, d\xi_1 d\xi_2 dt ds$$

and so, its expectation is

$$\mathbb{E}I(\mu_\varepsilon,\kappa) = \int \gamma(\varepsilon\xi_1)\bar{\gamma}(\varepsilon\xi_2)\mathbb{E}(e^{i(\xi_1 f(t)-\xi_2 f(s))})\kappa(t-s)\, d\xi_1 d\xi_2 dt ds\,.$$

Thus in order to get lower bounds on the dimension, we need estimates on $\mathbb{E}(e^{i(\xi_1 f(t)-\xi_2 f(s))})$. Plugging in the estimates tells us that the above expectation is uniformly finite for $\kappa(t) = |t|^{-\beta}$ for any $\beta < n - 2\alpha$.

As $\mu_\epsilon(B)$ is bounded independent of ϵ for any ball B, we get a weakly convergent subsequence $\mu_\epsilon \to \mu$. It is clear that μ is non-negative, supported on $L_0(f)$, and that $\mathbb{E}I(\mu,\kappa) < +\infty$. So we're left to prove that $\mu \neq 0$ almost surely. To this end, note that for any ball B of radius 1,

$$\mathbb{P}(\mu(B) \neq 0) \geqslant \frac{(\mathbb{E}(\mu(B)))^2}{\mathbb{E}(\mu(B)^2)}\,.$$

But now

$$\mathbb{E}(\mu_\epsilon(B)) = \int_B \int \gamma(\epsilon\xi)\mathbb{E}(e^{i\xi f(x)})\, d\xi dx = \int_B \int \gamma(\epsilon\xi) \prod \hat{F}(2^{-k\alpha}\xi\psi^i(2^k x+l))\, d\xi dx \geqslant C > 0\,,$$

for a constant C independent of the location of the ball B. That $\mathbb{E}(\mu(B)^2)$ is bounded above follows similarly to the computation in the previous section. So we conclude that $\mathbb{P}(\mu(B) \neq 0) > 0$ independent of B.

Now as $\mu(B_1(i))$ for $i \in 3\mathbb{Z}$ are independent events, by the Kolmogorov 0-1 law, we conclude that $\mu \neq 0$ almost surely.