Wavelets and Hilbert Bases

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In homage to A. P. Calderón
Translated by John Horváth

1. Introduction

Wavelet transforms appear implicitly in a famous work of A. P. Calderón [2]. It was rediscovered and made explicit by J. Morlet and his collaborators [4], [7], [8] as an efficient technique of numerical analysis allowing signal processing in connection with oil prospecting.

Wavelet transforms are similar to the Fourier transformation but the imaginary exponentials $\exp(ix \cdot \xi)$ indexed by the frequencies $\xi \in \mathbf{R}^n$ are replaced by the "wavelets" ψ_Q indexed by the collection of all the cubes $Q \subset \mathbf{R}^n$. These "wavelets" ψ_Q are all copies (by translation and change of scale) of the same regular function ψ, decreasing at infinity as strongly as possible. To be more precise, we choose ψ in the class $\mathcal{S}(\mathbf{R}^n)$ of L. Schwartz and if the cube Q is defined by $a_j \leq x_j \leq a_j + d$, $1 \leq j \leq n$, where $a = (a_1, \dots, a_n)$, then $\psi_Q(x) = d^{-n/2}\, \psi((x-a)/d)$. This means that ψ_Q is *localized* and *adjusted to the cube* Q.

The goal is to be able to write, by a judicious choice of ψ, any function f as a sum or as an integral of the $c_Q\, \psi_Q(x)$, where the coefficients c_Q are calculated like Fourier coefficients:

$$c_Q = \int_{\mathbf{R}^n} f(x)\, \overline{\psi}_Q(x)\, dx.$$

The authors mentioned above have obtained conditions on ψ allowing us to reconstitute f with the help of the *redundant information* given by the knowledge of all the "wavelet coefficients" c_Q.

Following a procedure made explicit by L. Carleson in [3], we propose to make this redundancy disappear by replacing the collection of all the cubes $Q \subset \mathbf{R}^n$ by that, denoted $\mathcal{Q}$, of the *dyadic cubes*. A dyadic cube Q is defined by two indices $j \in \mathbf{Z}$, $k \in \mathbf{Z}^n$, and one has

$$Q = \{x \in \mathbf{R}^n; 2^j x - k \in [0,1]^n\}. \tag{1.1}$$

The construction that will follow is, as in [3], an imitation of the Haar system of which we recall the definition.

We call $h^{(1)}(t)$ the function equal to 1 if $0 \leq t < 1/2$, to -1 if $1/2 \leq t < 1$, and to 0 elsewhere. On the other hand, $h^{(0)}(t)$ is the characteristic function of the interval

Translator's Note: The references have been updated, and a new version of Figure 1 was generated by Norbert Kaiblinger.

Authors' Note: As is the case for the paper "Uncertainty Principle, Hilbert Bases, and Algebras of Operators" reprinted preceding this one, the notion of multiresolution analysis is still missing from this paper, and so is any reference to Strömberg's work.

$[0,1]$. We denote by $E \subset \{0, 1\}^n$ the set of the $2^n - 1$ sequences $(\epsilon_1, \ldots, \epsilon_n)$ of zeros and ones, excluding the sequence $(0, 0, \ldots, 0)$. Then the Haar system for $L^2(\mathbf{R}^n)$ consists of the functions $h_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, defined by

$$h_Q^{(\epsilon)}(x) = 2^{nj/2}\, h^{(\epsilon_1)}(2^j x_1 - k_1) \cdots h^{(\epsilon_n)}(2^j x_n - k_n).$$

The Haar system is well adapted to the function spaces $L^p(\mathbf{R}^n)$, $1 < p < +\infty$, but does not permit the study of Sobolev and Besov spaces, or of the Hardy spaces $H^1(\mathbf{R}^n)$ of Stein and Weiss.

The generally admitted idea is that "to smooth the Haar system makes the orthogonality disappear." This is what appears reading [3] or [6]. We shall see, however, that there exists an exceptional choice of two functions $\psi^{(0)}$ and $\psi^{(1)}$ of $\mathcal{S}(\mathbf{R})$ such that the sequence of functions

$$\psi_Q^{(\epsilon)}(x) = 2^{nj/2}\, \psi^{(\epsilon_1)}(2^j x_1 - k_1) \cdots \psi^{(\epsilon_n)}(2^j x_n - k_n) \tag{1.2}$$

is a Hilbert basis of $L^2(\mathbf{R}^n)$ ($\epsilon \in E$, $Q \in \mathcal{Q}$).

This basis suits all the classical function spaces: Sobolev, Besov, Hardy, ..., spaces that translate isomorphically into sequence spaces. In particular, the coefficients of a function $f \in H^1(\mathbf{R}^n)$ with respect to the basis $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, and those of a function g in the dyadic version of H^1 with respect to the basis $h_Q^{(\epsilon)}$ are characterized by the same condition. This remark yields a very simple proof of Maurey's theorem (see Theorem 4 below).

In the fundamental identity

$$f(x) = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \langle f, \psi_Q^{(\epsilon)} \rangle\, \psi_Q^{(\epsilon)}(x) \tag{1.3}$$

we dispose of several levels of reading.

If $f \in L^2(\mathbf{R}^n)$, then we deal with the usual decomposition with respect to an orthonormal basis. The convergence is in the quadratic mean and almost everywhere.

If $f(x)$ and its derivatives of order 1 belong to $L^2(\mathbf{R}^n)$, then we have also convergence in $L^2(\mathbf{R}^n)$ of the termwise differentiated series. If more generally f and its derivatives up to order s ($s \in \mathbf{N}$) belong to $L^2(\mathbf{R}^n)$, then the series (1.3) will converge automatically with respect to the corresponding norm, that is, that of the Sobolev space H^s. In other words, the regularity of f accelerates the convergence (as this is the case for Fourier series but certainly not for the Haar system). Finally, we will show that this new algorithm permits a very simple writing of the paraproduct $\pi(a, f)$ of J. M. Bony [1].

2. Statement of the Fundamental Theorem

We begin by constructing special functions of one real variable. We denote by S the interval $[2\pi/3, 4\pi/3]$. Thus we have $2S = [4\pi/3, 8\pi/3]$, $S - 2\pi = -S = [-4\pi/3, -2\pi/3]$ and hence also $2\pi - S = S$. We denote by $\theta(t)$ an *odd* function of the real variable t, of class C^∞, with values in $[-\pi/4, \pi/4]$, equal to $\pi/4$ if $t \geq \pi/3$ (hence to $-\pi/4$ if $t \leq -\pi/3$). One defines the even function $\omega(t)$ by $\omega(t) = 0$ if $0 \leq t \leq 2\pi/3$ or if $t \geq 8\pi/3$, $\omega(t) = \pi/4 + \theta(t - \pi)$ if $2\pi/3 \leq t \leq 4\pi/3$, and $\omega(t) = \pi/4 - \theta(t/2 - \pi)$ if $4\pi/3 \leq t \leq 8\pi/3$.

The fundamental identities satisfied by $\omega(t)$ are $\omega(-t) = \omega(t)$, $\omega(2t) = \pi/2 - \omega(t)$, $t \in S$, and

$$\omega(t - 2\pi) = \omega(2\pi - t) = \frac{\pi}{2} - \omega(t), \quad t \in S.$$

By construction $\omega(t)$ is infinitely differentiable.

With the help of $\omega(t)$ one defines $\psi \in \mathcal{S}(\mathbf{R})$ by

$$\widehat{\psi}(t) = e^{-it/2} \sin \omega(t) = e^{-it/2}\, \widetilde{\omega}(t). \tag{2.1}$$

Thus we have

$$\psi(t) = \frac{1}{\pi} \int_0^\infty \cos\left[\left(t - \frac{1}{2}\right) s\right] \widetilde{\omega}(s)\, ds. \tag{2.2}$$

As J. Morlet made us observe, one can also start from an *odd* function $\omega_1(t)$ defined by $\omega_1(t) = -\omega(t)$ if $t \geq 0$ (and hence $\omega_1(t) = \omega(t)$ if $t \leq 0$). We associate with it $\psi_1 \in \mathcal{S}(\mathbf{R})$ defined by

$$\widehat{\psi_1}(t) = e^{-it/2} \sin \omega_1(t), \tag{2.3}$$

which implies

$$\psi_1(t) = \frac{1}{\pi} \int_0^\infty \sin\left[\left(t - \frac{1}{2}\right) s\right] \sin \omega(s)\, ds. \tag{2.4}$$

In the theorems that follow ψ can be replaced systematically by ψ_1. If $I \subset \mathbf{R}$ is the dyadic interval defined by $2^j x - k \in [0, 1]$, we define ψ_I by $\psi_I(x) = 2^{j/2}\, \psi(2^j x - k)$.

Then we have

Theorem 1 *The collection of wavelets $\psi_I(x)$ is a Hilbert basis of $L^2(\mathbf{R})$.*

We shall give the corresponding statement in $\mathbf{R}^n$ resuming the notation of the introduction and imitating the construction of the Haar system. The function ψ stays the same as above, and we associate with it a function $\varphi \in \mathcal{S}(\mathbf{R})$ having integral 1, defined by $\widehat{\varphi}(t) = \cos \omega(t)$ if $|t| \leq 4\pi/3$ and by $\widehat{\varphi}(t) = 0$ otherwise.

One sets $\psi^{(0)}(t) = \varphi(t)$, $\psi^{(1)}(t) = \psi(t)$ and defines $\psi^{(\epsilon)} \in \mathcal{S}(\mathbf{R}^n)$ by

$$\psi^{(\epsilon)}(x) = \psi^{(\epsilon_1)}(x_1)\, \psi^{(\epsilon_2)}(x_2) \cdots \psi^{(\epsilon_n)}(x_n),$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in E$; this means that $\epsilon_j = 0$ or 1 but the sequence consisting only of zeros is excluded.

Let us recall that a dyadic cube Q is defined by $j \in \mathbf{Z}$, $k \in \mathbf{Z}^n$ and is the set of the $x \in \mathbf{R}^n$ such that $2^j x - k \in [0, 1]^n$. We then set

$$\begin{aligned} \psi_Q^{(\epsilon)}(x) &= 2^{nj/2}\, \psi^{(\epsilon)}(2^j x - k) \\ &= 2^{nj/2}\, \psi^{(\epsilon_1)}(2^j x_1 - k_1) \cdots \psi^{(\epsilon_n)}(2^j x_n - k_n). \end{aligned} \tag{2.5}$$

With this notation, we intend to prove the following result.

Theorem 2 *The collection of functions $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, is a Hilbert basis of $L^2(\mathbf{R}^n)$.*

Before proving this result, let us introduce some notation. In spite of the fact that the choice of ψ is not unique, the functions $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, will be called the *wavelets*. Any series $\sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q)\, \psi_Q^{(\epsilon)}(x)$ is called a series of wavelets.

If $f(x)$ belongs to $L^2(\mathbf{R}^n)$ or, more generally, is a tempered distribution, the coefficients

$$\alpha(\epsilon, Q) = \int f(x)\, \psi_Q^{(\epsilon)}(x)\, dx \tag{2.6}$$

will be called the *wavelet coefficients*. An easy consequence of Theorem 2 is that one will always have the inversion formula

$$f(x) = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q)\, \psi_Q^{(\epsilon)}(x) \tag{2.7}$$

in the following sense: for every test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$ whose moments are all zero,

$$\langle f, u\rangle = \sum_{\epsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha(\epsilon, Q) \int_{\mathbf{R}^n} \psi_Q^{(\epsilon)}(x)\, u(x)\, dx.$$

Finally, the transformation that associates with the tempered distribution f the numerical sequence $\alpha(\epsilon, Q)$ will be called the *wavelet transform*. It plays the role of a local Fourier transformation (in fact the integration is extended over the whole $\mathbf{R}^n$, but there is a strong decay due to the rapid decrease of ψ). We split the proof of Theorem 2 into two parts.

First we shall prove that $\psi_Q^{(\epsilon)}$ is orthogonal to $\psi_{Q'}^{(\epsilon')}$ if $(Q, \epsilon) \neq (Q', \epsilon')$. Then we shall show that any function $f \in L^2(\mathbf{R}^n; dx)$ orthogonal to each of the $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, is necessarily zero.

The orthogonality relations follow from the corresponding properties in one real variable (which we will describe).

Lemma 1 *One has*

$$\int_{-\infty}^{\infty} \widehat{\varphi}(t)\, \overline{\widehat{\psi}}(2^{-j}t)\, e^{ik2^{-j}t}\, dt = 0$$

for all $j \in \mathbf{N}$ and all $k \in \mathbf{Z}$. Similarly one has

$$\int_{-\infty}^{\infty} \widehat{\psi}(t)\, \overline{\widehat{\psi}}(2^{-j}t)\, e^{ik2^{-j}t}\, dt = 0$$

for $j \geq 1$ and $k \in \mathbf{Z}$.

Let us begin by proving the first relation. If $j \geq 1$, the supports of $\widehat{\varphi}(t)$ and of $\widehat{\psi}(2^{-j}t)$ are disjoint.

If $j = 0$, one has $\widehat{\varphi}(t)\, \overline{\widehat{\psi}}(t) = e^{it/2} \sin\omega(t) \cos\omega(t)$ if $|t| \leq 4\pi/3$, and $= 0$ otherwise. The support of $\widehat{\varphi}\overline{\widehat{\psi}}$ is thus composed of the two intervals $S = [2\pi/3,\, 4\pi/3]$ and $S - 2\pi$. The identity $\omega(t - 2\pi) = \pi/2 - \omega(t)$ and the relation $e^{i\pi} = -1$ conclude the verification.

The proof of the second assertion is the same. If $j \geq 2$, the supports of $\widehat{\psi}(t)$ and of $\widehat{\psi}(2^{-j}t)$ are disjoint. It remains to consider the case $j = 1$ and one makes in the integral the change of variables $t = 2s$. Returning to the variable t, one observes that $\widehat{\psi}(2t)\widehat{\psi}(t) = e^{-it/2} \sin\omega(t) \cos\omega(t)$ if $|t| \leq 4\pi/3$, and $= 0$ elsewhere. We are brought back to the preceding case.

We shall now elucidate the case $j = 0$.

Lemma 2 *For every $k \in \mathbf{Z}$, $k \neq 0$ one has*

$$\int_{-\infty}^{\infty} |\widehat{\varphi}(t)|^2 \, e^{ikt} \, dt = \int_{-\infty}^{\infty} |\widehat{\psi}(t)|^2 \, e^{ikt} \, dt = 0.$$

To see this, we will prove that

$$\sum_{-\infty}^{\infty} |\widehat{\psi}(t + 2\pi j)|^2 = \sum_{-\infty}^{\infty} |\widehat{\varphi}(t + 2\pi j)|^2 = 1.$$

We shall limit ourselves to the first identity. The sum to be calculated defines a 2π-periodic function which it suffices to know on an interval of length 2π. We choose the interval $2\pi/3 \leq t \leq 8\pi/3$. If $2\pi/3 \leq t \leq 4\pi/3$, the only values of j that arise are $j = 0$ and $j = -1$. Because $\omega(t - 2\pi) = \pi/2 - \omega(t)$, the identity follows from $\sin^2 \omega(t) + \cos^2 \omega(t) = 1$. If $4\pi/3 \leq t \leq 8\pi/3$, one has to take $j = 0$ and $j = -2$, and one concludes in the same way.

Let us return to the orthogonality between the $\psi_Q^{(\epsilon)}$. One is led to calculate the integral

$$I(j, j', k, k', \epsilon, \epsilon') = \int_{\mathbf{R}^n} \exp[i(k2^{-j} - k'2^{-j'})\xi] \, \widehat{\psi}^{(\epsilon_1)}(2^{-j}\xi_1) \, \overline{\widehat{\psi}^{(\epsilon_1')}}(2^{-j'}\xi_1) \cdots \widehat{\psi}^{(\epsilon_n)}(2^{-j}\xi_n) \, \overline{\widehat{\psi}^{(\epsilon_n')}}(2^{-j'}\xi_n) \, d\xi_1 \cdots d\xi_n.$$

By symmetry we may assume that $j \leq j'$. One makes immediately the change of variables $2^{-j}\,\xi = u$ and is thereby reduced to the case $j = 0$, which we will suppose henceforth. If $j' \geq 1$, we call m a subscript such that $\epsilon_m = 1$ (such a subscript exists since $\epsilon \in E$). One integrates first with respect to the variable x_m, and Lemma 1 ensures that I is zero. If $j' = 0$ and $k \neq k'$, there exists a subscript m such that $k_m \neq k'_m$. One integrates first with respect to the variable x_m and I is zero, either because of the first assertion of Lemma 1 or because of Lemma 2.

Finally, the last case to consider is $j' = 0$, $k = k'$, and $\epsilon \neq \epsilon'$. Then there is a subscript m such that $\epsilon_m = 1$ and $\epsilon'_m = 0$ and the first assertion of Lemma 1 implies the nullity of the integral.

One verifies without effort that $\|\psi_Q^{(\epsilon)}\|_2 = 1$ for all ϵ and all Q.

3. The Sequence of the $\psi_Q^{(\epsilon)}$ is a Total Subset in $L^2(\mathrm{R}^n)$

We arrive at the most technical part of the proof of Theorem 2.

We will verify that $\langle f, \psi_Q^{(\epsilon)} \rangle = 0$ for all $\epsilon \in E$ and all $Q \in \mathcal{Q}$ implies $f = 0$.

For this we will write these relations using the Fourier transformation. Changing the notation as convenient, we have for a certain function $f \in L^2(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} f(\xi) \, \exp(ik2^{-j}\xi) \, \widehat{\psi}^{(\epsilon_1)}(2^{-j}\xi_1) \cdots \widehat{\psi}^{(\epsilon_n)}(2^{-j}\xi_n) \, d\xi_1 \cdots d\xi_n = 0 \tag{3.1}$$

for every $j \in \mathbf{Z}$, every $k \in \mathbf{Z}^n$, and every sequence $(\epsilon_1, \ldots, \epsilon_n) \neq (0, \ldots, 0)$ of zeros and ones.

We must deduce from this that $f = 0$. We proceed to the change of variables $2^{-j}\xi = u$. Then (3.1) becomes

$$\int_{\mathbf{R}^n} f_j(\xi)\, \exp(ik\xi)\, \widehat{\psi}^{(\epsilon_1)}(\xi_1) \cdots \widehat{\psi}^{(\epsilon_n)}(\xi_n)\, d\xi = 0 \text{ where } f_j(\xi) = f(2^j\xi). \tag{3.2}$$

It is classical that for a fixed j and all $k \in \mathbf{Z}^n$ (3.2) is equivalent to the identity

$$\sum_{k\in\mathbf{Z}^n} f_j(\xi + 2k\pi)\, \widehat{\psi}^{(\epsilon_1)}(\xi_1 + 2k_1\pi) \cdots \widehat{\psi}^{(\epsilon_n)}(\xi_n + 2k_n\pi) = 0. \tag{3.3}$$

This identity must be satisfied for all $j \in \mathbf{Z}$, all $\epsilon \in E$, and all $\xi \in \mathbf{R}^n$. The ideal situation would be if there existed a subset $\Omega \subset \mathbf{R}^n$ having the following properties: the translates $\Omega + 2k\pi$, $k \in \mathbf{Z}^n$ are pairwise disjoint; for every $\xi \in \mathbf{R}^n$ there exists a $j \in \mathbf{Z}^n$ such that $2^j\xi \in \Omega$; and finally, for every $\xi \in \Omega$ there exists $\epsilon \in E$ such that $\widehat{\psi}^{(\epsilon_1)}(\xi_1) \neq 0, \ldots, \widehat{\psi}^{(\epsilon_n)}(\xi_n) \neq 0$. Then (3.3) would imply that $f_j(\xi) = 0$ on Ω for all $j \in \mathbf{Z}$ and thus $f = 0$. Naturally this is not the case, but we will try to come as close to this ideal situation as possible.

To simplify the notation which follows, we shall restrict ourselves to dimension $n = 2$. Let us write again (3.3) in detail:

$$\sum_k \sum_\ell f_j(u + 2k\pi, v + 2\ell\pi)\, \widehat{\psi}(u + 2k\pi)\, \widehat{\psi}(v + 2\ell\pi) = 0, \tag{3.4}$$

$$\sum_k \sum_\ell f_j(u + 2k\pi, v + 2\ell\pi)\, \widehat{\varphi}(u + 2k\pi)\, \widehat{\psi}(v + 2\ell\pi) = 0, \tag{3.5}$$

$$\sum_k \sum_\ell f_j(u + 2k\pi, v + 2\ell\pi)\, \widehat{\psi}(u + 2k\pi)\, \widehat{\varphi}(v + 2\ell\pi) = 0. \tag{3.6}$$

The sums (3.4) to (3.6) define obviously $(2\pi\mathbf{Z})^2$-periodic functions, which we will analyze on a “fundamental domain.” First we shall suppose that

$$(u, v) \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]^2 = \Omega_1$$

and consider that (3.4), (3.5), and (3.6) are coupling $f_j(u, v)$ (which we try to calculate) to the undesirable values $f_j(u + 2k\pi, v + 2\ell\pi)$, which we will try to eliminate.

For $(u, v) \in \Omega_1$ we have necessarily $k = 0$ or $k = -1$ and $\ell = 0$ or $\ell = -1$. Setting $f_j = f_j(u, v)$, $g_j = f_j(u - 2\pi, v)$, $\widetilde{f}_j = f_j(u, v - 2\pi)$, and $\widetilde{g}_j = f_j(u - 2\pi, v - 2\pi)$, we have

$$\begin{aligned} &\sin\omega(u)\, \sin\omega(v)\, f_j - \cos\omega(u)\, \sin\omega(v)\, g_j \\ &\quad - \sin\omega(u)\, \cos\omega(v)\, \widetilde{f}_j + \cos\omega(u)\, \cos\omega(v)\, \widetilde{g}_j = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\cos\omega(u)\, \cos\omega(v)\, f_j + \sin\omega(u)\, \sin\omega(v)\, g_j \\ &\quad - \cos\omega(u)\, \cos\omega(v)\, \widetilde{f}_j - \sin\omega(u)\, \cos\omega(v)\, \widetilde{g}_j = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\sin\omega(u)\, \cos\omega(v)\, f_j - \cos\omega(u)\, \cos\omega(v)\, g_j \\ &\quad + \sin\omega(u)\, \sin\omega(v)\, \widetilde{f}_j - \cos\omega(u)\, \sin\omega(v)\, \widetilde{g}_j = 0. \end{aligned} \tag{3.9}$$

These three relations obviously do not allow us to calculate the four unknowns, which are f_j, g_j, $\widetilde{f}_j$, and $\widetilde{g}_j$.

We consider (3.4) for the point $(u', v') = (2u, 2v)$.

Then the sole values of k and ℓ that need be considered are $k = 0$ or $k = -2$, $\ell = 0$ or $\ell = -2$.

We observe that $f_j(u', v') = f_{j+1}(u, v)$, $f_j(u' - 4\pi, v') = f_{j+1}(u - 2\pi, v)$, etc, ... Shifting the subscript j by a unit, we therefore obtain the missing relation in the form

$$\begin{aligned} &\sin\omega(u)\, \sin\omega(v)\, f_j + \cos\omega(u)\, \sin\omega(v)\, g_j \\ &\quad + \sin\omega(u)\, \cos\omega(v)\, \widetilde{f}_j + \cos\omega(u)\, \cos\omega(v)\, \widetilde{g}_j = 0. \end{aligned} \tag{3.10}$$

The determinant of the four equations written is equal to 1, and we obtain thus $f_j = g_j = \widetilde{f}_j = \widetilde{g}_j = 0$ for every $j \in \mathbf{Z}$.

Let us denote now by Ω_2 the rectangle $\pi/3 \le u \le 2\pi/3$, $2\pi/3 \le v \le 4\pi/3$. We must observe that $\widetilde{\psi}(u) = 0$ on Ω_2 while $\widetilde{\varphi}(u) = 1$. The only relation pertinent for calculating $f_j(u, v)$ if $(u, v) \in \Omega_2$ is therefore (3.5). The only values of k and ℓ that really occur are $k = 0$ and $\ell = 0$ or $\ell = -1$.

We still set $f_j = f_j(u, v)$ and $\widetilde{f}_j = f_j(u, v - 2\pi)$ and have

$$f_j\, \sin\omega(v) - \widetilde{f}_j\, \cos\omega(v) = 0. \tag{3.11}$$

Obviously this sole relation does not permit us to calculate f_j and $\widetilde{f}_j$. But here again we consider the point $(2u, 2v)$ by which one writes (3.4) and (3.5). The only values of k or ℓ that occur are $k = 0$ and $k = -1$, $\ell = 0$ or $\ell = -2$. Let us set $\sigma_j = f_j(u, v)\, \cos\omega(v) + f_j(u, v - 2\pi)\, \sin\omega(v)$ and $\tau_j = f_j(u - \pi, v)\, \cos\omega(v) + f_j(u - \pi, v - 2\pi)\, \sin\omega(v)$ (having taken care to shift the subscripts by a unit).

Thus we have $\sigma_j \cos\omega(2u) + \tau_j \sin\omega(2u) = 0$ and $\sigma_j \sin\omega(2u) - \tau_j \cos\omega(2u) = 0$. These two relations imply $\sigma_j = 0$ or

$$f_j\, \cos\omega(v) - \widetilde{f}_j\, \sin\omega(v) = 0. \tag{3.12}$$

The relations (3.11) and (3.12) imply $f_j = \widetilde{f}_j = 0$ for every $j \in \mathbf{Z}$. Naturally, the same approach suits the other seven rectangles obtained by symmetry with respect to the coordinate axes or their bisectors, starting with Ω_2. Finally, one studies the rectangle Ω_3 defined by $0 \le u \le \pi/3$ and $2\pi/3 \le v \le 4\pi/3$. The only significant relation is again $f_j\, \sin\omega(v) - \widetilde{f}_j\, \cos\omega(v) = 0$. We use the same strategy as above, writing (after the necessary shift of j) the corresponding relation on the rectangle $2\Omega_3$ for the point $(2u, 2v)$. It follows that $f_j\, \cos\omega(v) - \widetilde{f}_j\, \sin\omega(v) = 0$, which yields $f_j = \widetilde{f}_j = 0$ on Ω_3. Here again we obtain the same conclusion for all the rectangles deduced from Ω_3 by all the symmetries of the problem.

Due to the examination of these three "fundamental domains," the whole annulus $2\pi/3 \le \sup(|u|, |v|) \le 8\pi/3$ is filled and $f(2^j u, 2^j v) = 0$ in this annulus. It follows immediately that $f = 0$ almost everywhere on $\mathbf{R}^2$.

4. Convergence of the Series of Wavelets in the Spaces of Test Functions or Distributions

Let $\mathcal{S}(\mathbf{R}^n)$ be the space of test functions equipped with the usual topology and $\mathcal{S}_0 \subset \mathcal{S}$ the subspace formed by the functions $f \in \mathcal{S}$ that verify $\int_{\mathbf{R}^n} x^\alpha f(x)\,dx = 0$ for all $\alpha \in \mathbf{N}^n$ (i.e., all its moments are zero). Then all the wavelets $\psi_Q^{(\epsilon)}$ belong to $\mathcal{S}_0$. If a series of wavelets converges in $\mathcal{S}$, it also converges in $\mathcal{S}_0$ to a function of $\mathcal{S}_0$. One proves easily that the converse is true.

Lemma 3 *If $f \in \mathcal{S}_0$, then the series of wavelets converges to f in the sense of the topology of $\mathcal{S}$.*

What happens, then, when f belongs to $\mathcal{S}$? It is easy to show that the series of wavelets converges to f uniformly on $\mathbf{R}^n$ and that the same is true for the differentiated series.

Let us now start with the function 1 (identically equal to 1 on the whole $\mathbf{R}^n$). Every coefficient (in wavelets) of 1 is 0. Why does one obtain an absurd numerical equality?

Let us consider the topological dual space of $\mathcal{S}_0(\mathbf{R}^n)$. This is the quotient space $\mathcal{S}'(\mathbf{R}^n)/\mathbf{C}[x]$ of the tempered distributions modulo the polynomials. If S is such a tempered distribution, then the corresponding series of wavelets $\sum_{Q\in\mathcal{Q}} \lambda_Q\, \psi_Q^{(\epsilon)}(x)$ converges in the following sense: for every function $f \in \mathcal{S}_0$ there is numerical convergence of the series obtained by integration against f. This is the reason why one obtains $1 = 0$ (modulo the constant functions).

5. Characterization of the Spaces $L^p(\mathbf{R}^n)$, $1 < p \leq +\infty$, BMO$(\mathbf{R}^n)$ and of its Predual $H^1(\mathbf{R}^n)$ by Transformation into Wavelets

Let E be a separable Banach space. A sequence $(e_k)_{k\in\mathbf{N}}$ of elements of E is a basis of E if every vector $x \in E$ can be written uniquely as $x = \sum_0^\infty \xi_k\, e_k$, where the ξ_k are scalars and where the partial sums of the written series converge to x in the norm of E. The basis is said to be unconditional if there exists a constant $C \geq 1$ such that for every $m \geq 1$ and every sequence ξ_k, $k \in \mathbf{N}$, of scalars one has $\| \sum_0^m \lambda_k\, \xi_k\, e_k\|_E \leq C\, \| \sum_0^m \xi_k\, e_k\|_E$ as soon as $\sup|\lambda_k| \leq 1$.

This means that the condition for a series $\sum_0^\infty \xi_k\, e_k$ to belong to E concerns only the sequence $|\xi_k|$, $k \in \mathbf{N}$, and that if a sequence satisfies this condition then automatically every other sequence satisfies it whose absolute values are dominated termwise by those of the first one.

The space BMO$(\mathbf{R}^n)$ is not separable and therefore does not possess an unconditional basis. We will replace it by the separable version VMO$(\mathbf{R}^n)$, which is the closure in the norm of BMO$(\mathbf{R}^n)$ of the space $\mathcal{S}(\mathbf{R}^n)$ of test functions.

Theorem 3 *The sequence $\psi_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$ is an unconditional basis for $L^p(\mathbf{R}^n; dx)$, $1 < p \leq +\infty$, VMO$(\mathbf{R}^n)$, and $H^1(\mathbf{R}^n)$.*

To prove it one constructs explicitly the operator T that transforms $\psi_Q^{(\epsilon)}$ into $\lambda(Q,\epsilon)\, \psi_Q^{(\epsilon)}$ if $|\lambda(Q,\epsilon)| \leq 1$. The distributional kernel of T is $K(x,y) = \sum_{\epsilon\in E} \sum_{Q\in\mathcal{Q}} \lambda(Q,\epsilon)$

$\psi_Q^{(\epsilon)}(x)\,\psi_Q^{(\epsilon)}(y)$. One obtains easily the Calderón-Zygmund estimates

$$|\partial_x^\alpha\,\partial_y^\beta K(x,y)| \le C(\alpha,\beta)\,|x-y|^{-n-|\alpha|-|\beta|} \tag{5.1}$$

for all $x \in \mathbf{R}^n$, all $y \neq x$, $\alpha \in \mathbf{N}^n$, and $\beta \in \mathbf{N}^n$.

On the other hand, T is obviously bounded on $L^2(\mathbf{R}^n)$ since it is diagonalized in an orthonormal basis. Finally T verifies in the sense of the theorem of David and Journé $T(1) = T^*(1) = 0$.

It follows that T is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and on the "boundary spaces" $H^1(\mathbf{R}^n)$ and $\mathrm{VMO}(\mathbf{R}^n)$.

It remains to write down explicit criteria of belonging to these function spaces. To this effect we recall the construction of the Haar basis of $L^2(\mathbf{R}^n; dx)$. One denotes by $\widetilde{\psi}(x)$ the function of a real variable equal to 1 on $[0,1/2[$, to -1 on $[1/2,1[$, and to 0 elsewhere. One calls $\widetilde{\varphi}$ the characteristic function of the interval $[0,1]$ (equal to 1 on this interval and to 0 elsewhere). One then constructs for all $\epsilon \in E$ and all $Q \in \mathcal{Q}$

$$\widetilde{\psi}_Q^{(\epsilon)} = 2^{nj/2}\,\widetilde{\psi}^{(\epsilon_1)}(2^j x_1 - k_1)\,\cdots\,\widetilde{\psi}^{(\epsilon_n)}(2^j x_n - k_n),$$

where $\widetilde{\psi}^{(0)} = \widetilde{\varphi}$, $\widetilde{\psi}^{(1)} = \widetilde{\psi}$.

Theorem 4 *The isometry $U\colon L^2(\mathbf{R}^n; dx) \to L^2(\mathbf{R}^n; dx)$ which associates $\widetilde{\psi}_Q^{(\epsilon)}$ with $\psi_Q^{(\epsilon)}$ extends to an isomorphism of $L^p(\mathbf{R}^n)$ onto itself if $1 < p < +\infty$, to an isomorphism of the Hardy space $H^1(\mathbf{R}^n; dx)$ of Stein and Weiss onto its dyadic version H_d^1, and to an isomorphism of the space $\mathrm{BMO}(\mathbf{R}^n; dx)$ of John and Nirenberg onto its dyadic version.*

This means that the (wavelet) coefficients $\lambda(Q,\epsilon)$ of $f \in L^p(\mathbf{R}^n)$ are characterized by the condition

$$\left(\sum_\epsilon \sum_Q |\lambda(Q,\epsilon)|^2 |Q|^{-1} \chi_Q(x)\right)^{1/2} \in L^p(\mathbf{R}^n)$$

where $1 < p < \infty$. One denotes by $|Q|$ the volume of Q and by $\chi_Q(x)$ the characteristic ("indicator") function of Q.

If $p = 1$, then this condition characterizes the coefficients of $f \in H^1(\mathbf{R}^n)$. Finally we will show, and this will be the beginning of the proof, that the wavelet coefficients $\lambda(Q,\epsilon)$ of $f \in \mathrm{BMO}(\mathbf{R}^n)$ are characterized by Carleson's condition

$$\sum_\epsilon \sum_{Q \subset R} |\lambda(Q,\epsilon)|^2 \le C|R| \tag{5.2}$$

(for every dyadic cube R, the summation being extended over all dyadic subcubes $Q \subset R$).

Let us first prove that (5.2) implies the convergence for the topology $\sigma(\mathrm{BMO}, H^1)$ of the series $\sum_\epsilon \sum_Q \lambda(Q,\epsilon)\,\psi_Q^{(\epsilon)}$ to a function $f \in \mathrm{BMO}$. To do this we disregard the index ϵ and reason on finite sums. Passing to the limit is routine once the inequality $\|\sum_Q \lambda(Q)\,\psi_Q(x)\|_{\mathrm{BMO}} \le C_0 \;\|\!|\; \lambda(Q) \;\|\!|$ is established for these finite sums, C_0 depends only on n, and $\|\!|\cdot\|\!|$ is the lower bound of the $\sqrt{C}$ in (5.2).

Let $\varphi \in \mathcal{D}(\mathbf{R}^n)$ be a function having integral 1 and supported by the unit cube $0 \le x_j \le 1$ $(1 \le j \le n)$.

We set $\varphi_Q(x) = 2^{nj}\,\varphi(2^j x - k)$ so that $\varphi_Q(x)$ is supported by the dyadic cube Q and has integral equal to 1. Then one considers the distribution

$$K(x,y) = \sum_{Q\in\mathcal{Q}} \lambda(Q)\,\psi_Q(x)\varphi_Q(y).$$

One has $|\lambda(Q)| \le (C|Q|)^{1/2}$, from which it follows immediately that $K(x,y)$ satisfies (5.1). Let us show that the operator T, whose distributional kernel is $K(x,y)$, is bounded on $L^2(\mathbf{R}^n)$. One has

$$\begin{aligned}\|T(f)\|_2 &= \Big\| \sum_{Q\in\mathcal{Q}} \lambda(Q)\Big(\int f\,\varphi_Q\Big)\,\psi_Q(x)\Big\|_2 \\ &= \Big(\sum_{Q\in\mathcal{Q}} |\lambda(Q)|^2 \Big|\int f\,\varphi_Q\Big|^2\Big)^{1/2} \\ &\le C_0\ \|\!|\ \lambda(Q)\ \|\!|\ \ \|f\|_2\end{aligned}$$

due to the dyadic version of Carleson's inequality stated in the following lemma.

Lemma 4 *Let p_Q, $Q\in\mathcal{Q}$, x_Q, $Q\in\mathcal{Q}$, be two arbitrary positive sequences, and $\omega(x)$, $x\in\mathbf{R}^n$, the maximal function defined by $\omega(x) = \sup_{Q\ni x} x_Q$. Let us assume that for every dyadic cube $R\in\mathcal{Q}$ we have $\sum_{Q\subset R} p_Q \le |R|$. Then one has $\sum_{Q\in\mathcal{Q}} p_Q\,x_Q \le \int_{R^n}\omega(x)\,dx$.*

Therefore one has continuity of $T\colon L^2(\mathbf{R}^n; dx) \to L^2(\mathbf{R}^n; dx)$.

This, combined with (5.1), implies $T(1)\in \mathrm{BMO}$, which is in fact what we tried to prove.

Conversely let us assume that f belongs to BMO.

We denote by R a dyadic cube and by R_j, $j\ge 1$, the (nondyadic) cubes with the same center as R whose sides are 2^j times those of R. We decompose, as usual, f in the series

$$f = c(R) + f_0(x) + f_1(x) + \cdots + f_j(x) + \cdots,$$

where $c(R) = m_R f$ is the mean of f on R and where $f_0(x) = f(x) - c(R)$ on the cube R, $f_0(x) = 0$ elsewhere, and $f_j(x) = f(x) - c(R)$ on $R_j \setminus R_{j-1}$, $f_j(x) = 0$ elsewhere. The condition $f\in\mathrm{BMO}$ implies $|m_{R_{j-1}} f - m_{R_j} f| \le C\|f\|_{\mathrm{BMO}}$, hence $|m_{R_j} f - m_R f| \le C\,j\,\|f\|_{\mathrm{BMO}}$. Finally, $\|f_j\|_2 \le C(1+j)\,|R_j|^{1/2}\,\|f\|_{\mathrm{BMO}}$. One then writes $\langle f,\psi_Q\rangle = \sum_0^\infty \langle f_j,\psi_Q\rangle$.

For $j\ge 2$, one has

$$\langle f_j,\psi_Q\rangle \le \|f_j\|_2 \left(\int_{R_j\setminus R_{j-1}} |\psi_Q(x)|^2\,dx\right)^{1/2} \le C_m\,\|f_j\|_2 \left(\frac{|Q|}{|R_j|}\right)^m$$

for every $m\ge 1$ (due to the rapid decay of ψ and the geometric condition $Q\subset R$).

For $j=0$ or $j=1$, one uses Plancherel's identity and one has

$$\sum_Q |\langle f_j,\psi_Q\rangle|^2 = \|f_j\|_2^2 \le C\,|R|\,\|f\|_{\mathrm{BMO}}^2.$$

One concludes observing that

$$\sum_{Q\subset R}\sum_{j\geq 2} j^2\,\frac{|Q|^2}{|R_j|} = c_n\,|R|.$$

The end of the proof of Theorem 3 is now obvious. In fact it is classical that the dyadic BMO space is also characterized by (5.2) by decomposing the functions in the Haar system. Thus U establishes an isomorphism between these two spaces. By duality, U (which is its own transpose) defines an isomorphism between the usual Hardy space $H^1(\mathbf{R}^n)$ and its dyadic version. By interpolation U is an isomorphism on all the spaces $L^p(\mathbf{R}^n)$, $1 < p < +\infty$.

6. Characterization of the Homogeneous Hölder Spaces C^r, $r > 0$, of the Sobolev Spaces H^s and of the Besov Spaces $B_q^{s,p}$

Let us recall that if $0 < r < 1$, then f belongs to the homogeneous Hölder space C^r when $|f(x) - f(y)| \leq C\,|x-y|^r$ for all $x \in \mathbf{R}^n$ and all $y \in \mathbf{R}^n$. If $r = 1$, then we agree to replace C^r by the Zygmund class defined by $|f(x+y)+f(x-y)-2f(x)| \leq C|y|$ ($x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$).

If $r = m + s$, $0 < s \leq 1$, $m \in \mathbf{N}$, then one writes $f \in C^r$ when $\partial^\alpha f \in C^s$ for all $\alpha \in \mathbf{N}^n$ such that $|\alpha| = m$.

The Sobolev space H^s, $s \in \mathbf{R}$, is defined by

$$\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2\,(1+|\xi|^2)^s\,d\xi < \infty,$$

and finally the homogeneous Besov spaces $B_q^{s,p}$ are characterized by the Littlewood-Paley decomposition. One starts with a function $\theta \in \mathcal{D}(\mathbf{R}^n)$, zero in a neighborhood of 0 and such that $\sum_{-\infty}^{\infty} |\theta(2^j\xi)| \geq 1$ for every nonzero $\xi \in \mathbf{R}^n$. Then a tempered distribution f, modulo the polynomials, belongs to $B_q^{s,p}$ if and only if $\|\Delta_j(f)\|_p \leq 2^{-js}\,\epsilon_j$, where $\epsilon_j \in \ell^q(\mathbf{Z})$ and where $\mathcal{F}(\Delta_j(f))(\xi) = \theta(2^{-j}\xi)\,(\mathcal{F}f)(\xi)$ denoting by $\mathcal{F}$ the Fourier transform (acting on the tempered distribution f). We denote by $\|\cdot\|_p$ the norm in $L^p(\mathbf{R}^n; dx)$.

This space $B_q^{s,p}$ does not depend on the choice of θ. One has $B_\infty^{s,\infty} = C^s$ (homogeneous Hölder space) and $H^s = L^2 \cap B_2^{s,2}$ for $s \geq 0$.

Theorem 5 *A tempered distribution f (modulo the polynomials) belongs to $B_q^{s,p}$ if and only if $\sup_{\epsilon\in E} |\langle f, \psi_Q^{(\epsilon)}\rangle| = \alpha(k,j)$ satisfies*

$$\left(\sum_{j=-\infty}^{\infty}\left\{\left(\sum_{k\in\mathbf{Z}^n}(\alpha(k,j))^p\right)^{1/p} 2^{j(s+n(1/2-1/p))}\right\}^q\right)^{1/q} < +\infty \tag{6.1}$$

(with the usual changes if $p = +\infty$ or $q = +\infty$).

Let us first show the direct part. To simplify the writing we restrict ourselves to dimension 1.

Then the function $\overline{\widehat{\psi}}$ which serves to define the wavelets will be our function θ and one has

$$\langle f, \psi_Q \rangle = 2^{-j/2}(2\pi)^{-1} \int \widehat{f}(\xi)\, e^{i2^{-j}k\xi}\, \overline{\widehat{\psi}}(2^{-j}\xi)\, d\xi = 2^{-j/2}\, \Delta_j(f)(2^{-j}k).$$

To conclude one uses the evident sampling lemma.

Lemma 5 *There exists a constant $C = C(n)$ such that for every $R > 0$, every $p \in [1, +\infty]$, and every function $f \in L^p(\mathbf{R}^n; dx)$ whose Fourier transform is carried by the ball $|\xi| \le 20R$, one has $(\sum_{k\in\mathbf{Z}^n} R^{-n}|f(kR^{-1})|^p)^{1/p} \le C(n)\, \|f\|_p$.*

Naturally 20 can be replaced (changing, if necessary, the constant $C(n)$) by another constant. For us $20 \ge 8\pi/3$.

Thus one has for every $j \in \mathbf{Z}$

$$\left(2^{-nj}|\Delta_j f(2^{-j}k)|^p\right)^{1/p} \le C(n)\, \|\Delta_j(f)\|_p,$$

which obviously implies (6.1).

Conversely, assume that $\alpha(k,j)$ satisfies (6.1) and let us show that

$$f(x) = \sum\sum \alpha(k,j)\, 2^{j/2}\, \psi(2^j x - k)$$

belongs to $B_q^{s,p}$. One tests this belonging on the unit ball of the dual $B_{q'}^{-s,p'}$ for which one uses the direct part. The details are left to the reader.

7. Calculation of the Partial Sums of a Wavelet Expansion

Let f be a tempered distribution (modulo the polynomials). What can one say of the difference

$$g(x) = f(x) - \sum_{\epsilon\in E}\sum_{|Q|\le 1} \langle f, \psi_Q^{(\epsilon)}\rangle\, \psi_Q^{(\epsilon)}\ ?$$

It seems to be intuitive that the small cubes are responsible for the small details of $f(x)$. These small details are very important or significant if $f(x)$ is very irregular. So that our difference $g(x)$ should be infinitely differentiable.

Let us denote by $\mathcal{D}_Q^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}$, the orthogonal projection operator onto the vector $\psi_Q^{(\epsilon)}$ of the Hilbert space $L^2(\mathbf{R}^n; dx)$. Similarly denote by $\mathcal{E}_Q$, $Q \in \mathcal{Q}$, the orthogonal projection operator onto the vector φ_Q. The functions φ and ψ are defined by Theorem 2.

Let us call $\mathcal{Q}_j$ the collection of dyadic cubes with side 2^{-j} and set

$$\mathcal{D}_j = \sum_{\epsilon\in E}\sum_{Q\in\mathcal{Q}_j} \mathcal{D}_Q^{(\epsilon)} \quad \text{and similarly} \quad \mathcal{E}_j = \sum_{Q\in\mathcal{Q}_j} \mathcal{E}_Q.$$

One has the following remarkable identity:

Theorem 6 *For every $j \in \mathbf{Z}$, $\mathcal{D}_j = \mathcal{E}_{j+1} - \mathcal{E}_j$.*

Let us observe that $\mathcal{E}_j$ is a "regularizing version" of the conditional expectation operator with respect to the σ-algebra $\mathcal{F}_j$ generated by the cubes $Q \in \mathcal{Q}_j$. In fact, this conditional expectation is $E_j(f) = \sum_{Q \in \mathcal{Q}_j} \chi_Q \, 2^{nj} \langle f, \chi_Q \rangle$ and $\mathcal{E}_j$ is given by the same identity with the difference that $2^{nj}\,\chi_Q$ is replaced by the "C^∞ version" φ_Q of the normalized characteristic function of the cube Q.

Let us give the structure of the proof of Theorem 6 for dimension 1. One denotes by Δ_j the operator of convolution with $2^j\,\psi(2^j x)$ and by M_j the operator of pointwise multiplication by $\exp(2\pi i 2^j x)$. Then a harmless application of the Poisson summation formula shows that

$$\mathcal{D}_j = \Delta_j \Delta_j^* + \Delta_j M_j \Delta_j^* + \Delta_j M_{j+1} \Delta_j^* + \Delta_j M_j^* \Delta_j^* + \Delta_j M_{j+1}^* \Delta_j^*.$$

One denotes by S_j the operator of convolution with $2^j \varphi(2^j x)$, and one then has the following remarkable identities:

$$\begin{aligned} \Delta_j \Delta_j^* &= S_{j+1} S_{j+1}^* - S_j S_j^*, \\ \Delta_j M_j \Delta_j^* &= -S_j M_j S_j^*, \\ \Delta_j M_{j+1}^* \Delta_j^* &= S_{j+1} M_{j+1} S_{j+1}^*. \end{aligned}$$

On the other hand, one performs for $\mathcal{E}_j$ the calculation we sketched for $\mathcal{D}_j$ and obtains $\mathcal{E}_j = S_j S_j^* + S_j M_j S_j^* + S_j M_j^* S_j^*$. It follows from all this that

$$\mathcal{D}_j = \mathcal{E}_{j+1} - \mathcal{E}_j.$$

Let us pass to the case of higher dimensions. For instance, if $n = 2$ one has, keeping the notation of dimension 1,

$$\mathcal{E}_{j+1} \otimes \mathcal{E}_{j+1} - \mathcal{E}_j \otimes \mathcal{E}_j = \mathcal{D}_j \otimes \mathcal{D}_j + \mathcal{D}_j \otimes \mathcal{E}_j + \mathcal{E}_j \otimes \mathcal{D}_j,$$

and these three terms correspond to the three wavelets $\psi_0^{(\epsilon)}$, $\epsilon \in E$, $Q \in \mathcal{Q}_j$, necessary to obtain our basis in dimension 2.

Thus Theorem 6 is proved, and these remarks yield a new proof of Theorem 2.

Returning to $g(x)$, we then have

$$\begin{aligned} g(x) &= f(x) - \sum_{j \geq 0} \mathcal{D}_j(f) \\ &= f(x) - \sum_{j \geq 0} (\mathcal{E}_{j+1}(f) - \mathcal{E}_j(f)) \\ &= \mathcal{E}_0(f) \\ &= \sum_{k \in \mathbf{Z}^n} c_k \, \varphi(x - k), \end{aligned}$$

where the coefficients c_k are given by $c_k = \int f(u)\, \varphi(u - k)\, du$.

This means that $f(x)$ and $g(x)$ differ by a trivial error term.

8. Paraproducts and Wavelet Transforms

The function φ is the same as in Theorem 2. The operator S_j is the convolution with $2^{nj}\varphi(2^j x)$. One sets $\Delta_j = S_{j+1} - S_j$ and finally

$$\pi(a,f) = \sum_{-\infty}^{\infty} S_{j-3}(a)\,\Delta_j(f).$$

One assumes $a(x) \in L^\infty \cap C^r$ and one reasons modulo the r-regularizing operators, that is, those which transform H^s into H^{s+r} for all $s \in \mathbf{R}$. The error terms are analyzed by the following lemma.

Lemma 6 *Let $f_j(x)$, $j \in \mathbf{Z}$, be functions in $L^2(\mathbf{R}^n)$ whose Fourier transforms $\widehat{f_j}$ satisfy, for two constants $R_2 > R_1 > 0$,*

$$\widehat{f_j}(\xi) = 0 \quad \text{if} \quad |\xi| \le R_1 2^j \quad \text{or if} \quad |\xi| \ge R_2 2^j. \tag{8.1}$$

Then for every $s \in \mathbf{R}$ there exists a constant $C = C(R_1, R_2, s, n)$ such that

$$\left\| \sum_{-\infty}^{\infty} f_j(x) \right\|_{H^s} \le C \left(\sum_{-\infty}^{\infty} \|f_j\|_2^2 \,(1+4^j)^s \right)^{1/2}. \tag{8.2}$$

Using this lemma one begins by replacing $S_{j-3}(a)$ by $S_{j-10}(a)$. Following that, one calculates $\pi(a, \psi_Q^{(\epsilon)})$ when Q is defined by $2^m - k \in [0,1]^n$. Then $\Delta_j(\psi_Q^{(\epsilon)}) = 0$ unless $|m-j| \le 2$.

One defines (by linearity) the operator R_a by

$$R_a(\psi_Q^{(\epsilon)}) = \pi(a, \psi_Q^{(\epsilon)}) - S_{m-10}(a)\,\psi_Q^{(\epsilon)}$$

and, using our lemma and the characterization by series of wavelets of the Sobolev spaces, one shows that R_a is r-regularizing.

One can then state

Theorem 7 *If $r > 0$ and $a(x) \in C^r \cap L^\infty$, the paraproduct $\pi(a,f)$ is defined by linearity by*

$$\pi(a, \psi_Q^{(\epsilon)}) = S_Q(a)\,\psi_Q^{(\epsilon)}$$

where, by an abuse of language, $S_Q(a) = S_{j-10}(a)$ when

$$Q = \{x \in \mathbf{R}^n; 2^j x - k \in [0,1]^n\};$$

if furthermore $0 < r < 1$, one has, modulo an r-regularizing operator,

$$\pi(a, \psi_Q^{(\epsilon)}) = a(2^{-j}k)\,\psi_Q^{(\epsilon)}.$$

In other words, the paraproduct is diagonalized in the Hilbert basis of the wavelets.

9. Analysis of the Earlier Works on the Subject

The "continuous version" of the transformation into wavelets has a long history going back to the works of A. Calderón and his collaborators. The discrete version appears in 1980 when L. Carleson proves that the Haar system, smoothed correctly, is an unconditional basis of the space $H^1(\mathbf{T})$. Naturally Carleson's wavelets do not form a Hilbert basis, but there exists an associated biorthogonal system.

Another process was introduced by Frazier and Jawerth [6] and, independently, in [4] and [9].

The question is to exhibit ψ in $\mathcal{S}(\mathbf{R}^n)$ so that one has, for everyfunction $f \in L^2(\mathbf{R}^n; dx)$,

$$f(x) = \sum_{Q \in \mathcal{Q}} \langle f, \psi_Q \rangle \, \psi_Q(x) \tag{9.1}$$

without uniqueness.

It is sufficient for this that the Fourier transform of ψ be supported by the cube $-\pi \leq x_j \leq \pi$, $1 \leq j \leq n$, and that one has, for all $\xi \neq 0$,

$$1 = \sum_{-\infty}^{\infty} |\widehat{\psi}(2^j \xi)|^2. \tag{9.2}$$

If one has uniqueness in (9.1), then the functions $\psi_Q(x)$ are automatically orthogonal but this is excluded by the condition on the support of $\widehat{\psi}$.

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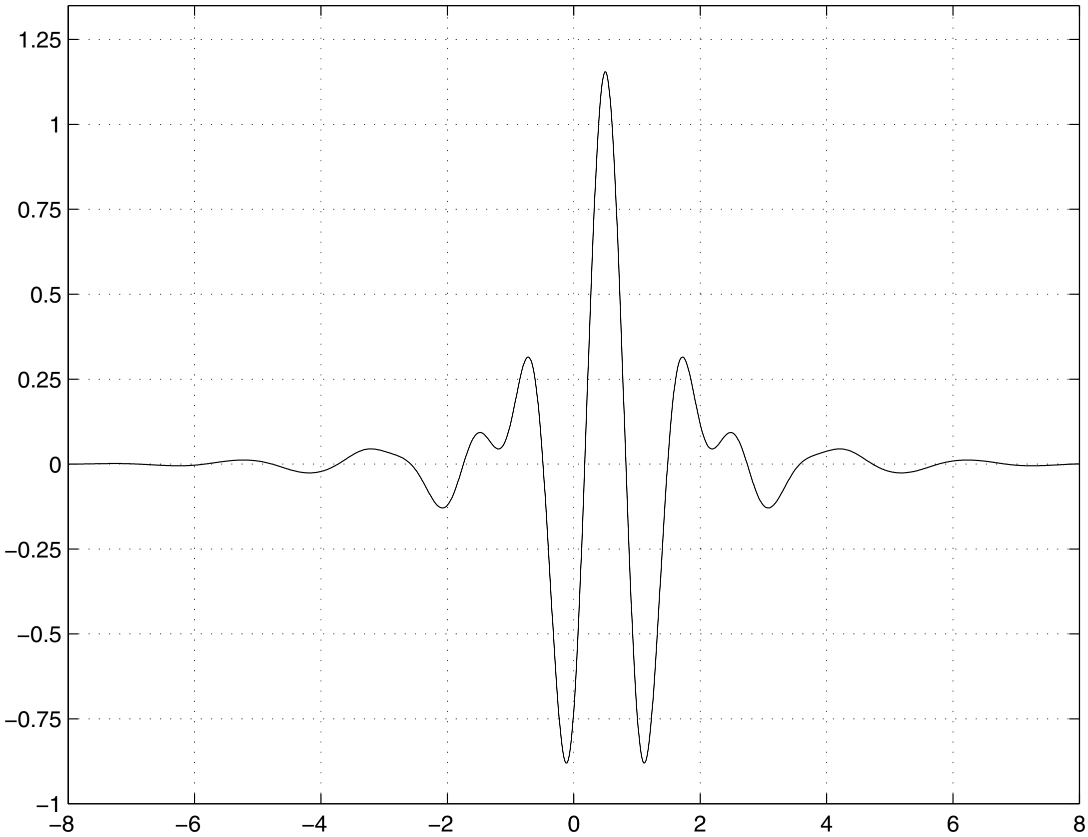

Figure 1. Wavelet