D-MATH | Commutative Algebra | HS 2016
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Exercise Sheet 1

Radical ideals, decomposition, Zariski topology

Exercises 2 and 6 are taken from the book *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. (a) Show that if $\mathfrak{a}$ is an ideal in a ring R and $\operatorname{Rad}(\mathfrak{a})$ its radical ideal, then $V(\mathfrak{a}) = V(\operatorname{Rad}(\mathfrak{a}))$.
   (b) Show that a proper ideal $\mathfrak{p} \subsetneqq R$ is a prime ideal if and only if, for any ideals $\mathfrak{a}, \mathfrak{b} \subset R$ with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, we have $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.
   (c) Show that every prime ideal is radical. Find an example which shows that the converse is *not* true.

2. Let X be a topological space. Show that
   (a) For any irreducible subspace Y of X, the closure $\overline{Y}$ of Y in X is irreducible.
   (b) Every irreducible subspace of X is contained in a maximal irreducible subspace.
   (c) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X.
   (d) What are the irreducible components of a Hausdorff space?

3. Determine the ideal in $\mathbb{R}[\underline{X}]$ of
   (a) the union of the three coordinate axes in $\mathbb{R}^3$,
   (b) the union of the lines containing the twelve edges of the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$,
   (c) the set $\{(n, e^n) \mid n \in \mathbb{Z}^{\geqslant 0}\}$ in $\mathbb{R}^2$.

4. Compute the irreducible components of $V(XZ - Y^2, X^3 - YZ)$ in $\mathbb{C}^3$.

*5 Show that every ring homomorphism $\varphi\colon R \to R'$ induces a continuous map $\operatorname{Spec} R' \to \operatorname{Spec} R$, $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

6. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of a ring R. Show that

(a) the set $\{\mathfrak{p}\}$ is closed in $\operatorname{Spec} R$ if and only if $\mathfrak{p}$ is maximal. In that case we call $\mathfrak{p}$ a *closed point*.

(b) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

(c) $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \iff \mathfrak{p} \subset \mathfrak{q}$.

(d) $\operatorname{Spec} R$ is a T_0-space, i.e., for any distinct points $\mathfrak{p}, \mathfrak{q}$ of $\operatorname{Spec} R$, there exists a neighborhood of $\mathfrak{p}$ which does not contain $\mathfrak{q}$, or a neighborhood of $\mathfrak{q}$ which does not contain $\mathfrak{p}$.